ERRATUM TO *A NEW INEQUALITY FOR THE GREEN'S FUNCTION*

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The paper [1] gave a short elementary derivation of the best-possible value of Koebe's constant, a derivation using geometric methods free of the arithmetic reasoning characteristic of "hard analysis." Professor Albert Pfluger has informed me that the inequality in my Lemma 3.1 must be reversed. This note contains a revised version of that lemma and of the proof of Theorem 3.2, which proof depends upon that lemma.

LEMMA (3.1)* (THE PRELIMINARY INEQUALITY). *The simple closed curve J_r in the upper half plane which is mapped by the function $F_r(w) = w^2+r$ onto the circle whose radius is $r>0$ and whose center is the origin separates the sphere into two domains: E_r, the exterior one, and I_r, the interior one. If e is any point of E_r then E_r admits a Green's function U_r (with pole at e) which is continuous on $(E_r \cup J_r)-(e)$ and which vanishes on J_r. Moreover, if $-e$ is in I_r and if $0<r<1$ then the inequality*

$$U_r(w) \leqq \ln \left| \frac{w/(r)^{1/2} + e}{w - e} \right|$$

holds for all w in E_r.

In [1], Lemma 3.1 claimed that

$$U_r(w) \leqq \ln \left| \frac{w + e}{w - e} \right|.$$

Since the right-hand member is the Green's function of that domain which contains e and is complementary to the line $|w+e| = |w-e|$ and since that domain is contained within E_r when, e.g., $e = -i$ and $r=1$ (the case used in Theorem 3.2), the lemma's claim was false.

PROOF OF (3.1)*. The existence of U_r and its property of vanishing on J_r are proved in [1]. The rest of the proof of (3.1)* proceeds as follows. Let G_r be defined by the rule

$$G_r(w) = \frac{w - e}{w/(r)^{1/2} + e}.$$

Evidently $G_r(e) = 0$ and $G_r(-er^{1/2}) = \infty$, so that $\ln |G_r|$ is harmonic

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on the sphere except at the points e and $-er^{1/2}$. Therefore the function $H_r = U_r + \ln|G_r|$ is harmonic on E_r except possibly at e and $-er^{1/2}$. By definition of the Green's function H_r is harmonic at e. Choosing $-e$ in I_r (whence $|F_r(e)| = |e^2+1| < 1$) and choosing $0 < r < 1$, it follows that $|F_r(-er^{1/2})| < 1$ so that $-er^{1/2}$ is in I_r and so is *not* in E_r. Therefore, H_r is harmonic on all of E_r and so is bounded there by the greatest of its boundary values. Since U_r vanishes on J_r one has

$$H_r(w) \leqq M(r) = \operatorname{lub}\{\ln|G_r(w)| : w \in J_r\}$$

for all w in E_r, provided $0 < r < 1$ and $-e$ is in I_r. [At this point in the proof of (3.1) in [1] the point $-e$, which there corresponded to the present $-er^{1/2}$, could not be guaranteed to belong to I_r for all sufficiently small r; nevertheless this inequality was used as r approached 0 and error was born.]

If a point w in E_r is chosen and if $r' < r$ then w is also in $E_{r'}$ and

$$H_r(w) = [U_r(w) - U_{r'}(w)] + \ln\left|\frac{u/(r)^{1/2} + e}{w(r')^{1/2} + e}\right| + H_{r'}(w).$$

The first bracketed term is not positive because the Green's function is an increasing functional of domain and $E_r \subset E_{r'}$. The logarithmic term is, for r' sufficiently close to zero, also not positive. Thus, for r' close to zero,

$$H_r(w) \leqq M(r').$$

Because the curves $J_{r'}$ shrink uniformly to the origin as r' approaches zero one may conclude that

$$\lim_{r' \to 0} M(r') = \lim_{r' \to 0;\, w \to 0} \ln|G_{r'}(w)| = \ln 1 = 0.$$

Thus $H_r \leqq 0$ on E_r, as required by (3.1)*.

The statement of Theorem 3.2 as it appears in [1] is (substantially) correct, but is repeated here for the reader's convenience. Its proof is then adapted to the use of (3.2)*.

THEOREM 3.2 (THE FINAL INEQUALITY). *Let D be a domain in the sphere with a boundary component K which contains at least two different points, p' and q, let d be a point of D, and let T be the rational fractional transformation which sends d, p, q respectively onto 0, 1, ∞, where $1 = T(p)$ is the point in $T(K)$ closest to $0 = T(d)$. Then D admits a Green's function U with pole at d such that $U(t) \to 0$ as $t \to p$, t in D. Moreover, if H is defined by the rule*

$$H(z) = U(T^{-1}(z)) + \ln |z|$$

then H is harmonic at $z = T(d) = 0$ and

$$H(0) \leqq \ln 4.$$

PROOF. Let T_r denote the rational fractional transformation which sends d, p, q respectively onto $0, r, \infty$, where $0 < r < 1$. It is convenient to denote $T_r(A)$ by $A(r)$ for any set or point A. In the notation of (3.2)*, the function F_r sends I_r onto L_r, the open disk of radius r and center at the origin. If I_r^* denotes the reflection in the real axis of I_r then $F_r(I_r^*) = L_r$ also. Let C_r denote the complement in the sphere of $K(r)$; one checks that C_r is a simply connected domain containing $D(r)$ which in turn contains L_r by choice of p. Since C_r contains neither r nor ∞, it follows that F_r is locally 1-1 on $F_r^{-1}(C_r)$. Using the simple connectedness of C_r one may construct (by Lemma 2.4 of [1]) two single-valued branches, g_r and h_r of F_r^{-1}. The functions g_r and h_r are analytic homeomorphisms defined on C_r with $g_r(0) = -ir^{1/2}$ and $h_r(0) = ir^{1/2}$. Evidently g_r sends L_r onto I_r^* and h_r sends L_r onto I_r. Since F_r is single-valued, this implies that $g_r(C_r)$ is disjoint from $I_r = h_r(L_r)$, so that its complement in the sphere contains an open set. By Lemma 2.2 of [1], the domain $g_r(C_r)$ admits a Green's function so that C_r and $D(r)$ do also. Let, therefore, u_r denote the Green's function for $D(r)$ with pole at the origin.

With $e = g_r(0)$, it follows (in the notation of (3.2)*) that $u_r \leqq U_r(g_r)$ on $D(r)$. Taking inverses, with $F_r = g_r^{-1}$ on $g_r(C_r)$, one has

$$u_r(F_r(w)) \leqq U_r(w) \leqq \ln \left| \frac{w/(r)^{1/2} + e}{w - e} \right|,$$

according to (3.2)*. If one defines H_r by the rule

$$H_r(z) = u_r(z) + \ln |z|$$

then H_r is harmonic on $D(r)$ and

$$\begin{aligned} H_r(F_r(w)) &\leqq \ln \left| \frac{w/(r)^{1/2} + e}{w - e} \right| + \ln |F_r(w)| \\ &\leqq \ln |w/(r)^{1/2} + e| + \ln \left| \frac{F_r(w)}{w - e} \right|. \end{aligned}$$

Because $F_r(e) = 0$, the ratio $F_r(w)/(w-e)$ approaches $F_{r'}(e) = 2e$ as $w \to e$. It follows that

$$H_r(F_r(e)) = H_r(0) \leqq \ln |-i - i(r)^{1/2}|, + \ln |-2i(r)^{1/2}|.$$

Since $u_r = U(T_r^{-1})$ it is clear that $H = H_1$. Since $T_r = rT_1$, it follows that $T_r^{-1}(z) = T_1^{-1}(z/r)$ so that $H_r(z) = H_1(z/r)$. Taking the limit as r, $0 < r < 1$, approaches one, one has

$$H(0) = \lim_{r\to 1} H_r(0) \leqq \ln 2 + \ln 2 = \ln 4,$$

as required.

Bibliography

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SOME DIFFERENTIAL GEOMETRIC PROPERTIES OF SUBMANIFOLDS OF EUCLIDEAN SPACES

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1. **Introduction.** The translation theorem of Hopf and Voss [3] has been generalized by Hsiung [4] and Voss [7] to hypersurfaces, by Hsü [6] to other elementary transformations, and these results were obtained for hypersurfaces by Aeppli [1]. In the first sections of this paper, these results will be generalized to n-dimensional manifolds in $(n+m)$-dimensional Euclidean space. In the final section, a condition will be given under which a submanifold of Euclidean space is a submanifold of a hypersphere, extending a result of Hsiung [5].

All manifolds mentioned will be assumed to be compact, connected, orientable, n-dimensional ($n \geqq 2$) manifolds with closed boundaries (empty or of dimension $n-1$) differentiably imbedded in an $(n+m)$-dimensional Euclidean space E^{n+m} ($m \geqq 1$). The notation adopted will be essentially that of Hsiung [4]. The following conventions will be adopted for indices:

$$\begin{aligned} \alpha, \beta, \cdots &= 1, \cdots, n+m, \\ \lambda, \mu, \cdots &= 1, \cdots, n+m-1, \\ a, b, \cdots &= 1, \cdots, m, \\ i, j, \cdots &= 1, \cdots, n, \\ r, s, \cdots &= 0, \cdots, n. \end{aligned}$$

Considerable use will be made of a vector product like that defined by Hsiung [5]. Namely, if $I_1, \cdots, I_{n+m}$ are a fixed frame of mutually

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