AN INEQUALITY FOR CERTAIN PENCILS OF PLANE CURVES

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1. **Introduction.** In this paper we use a method similar to one used by Engel in [1] in order to get an inequality relating the degree of the generic curve of a rational pencil of plane curves with the order of its divisor of singularities at a base point of the pencil satisfying some special conditions. In [1] Engel uses a particular case of this relation to give a new proof of the following theorem of Jung:

Consider the affine plane over the field k of complex numbers. Then any entire Cremona transformation is a product of linear transformations and transformations of the following type:

$$x' = x,$$
$$y' = y + cx^n$$

where $c \in k$ and n is a positive integer.

For the convenience of the reader we sketch how one can obtain a simple proof of the preceding result using our Corollary A in §7.

Our Lemma B is an abstraction and generalization of a lemma of Jung [2], and is proved in more generality than needed here for further reference.

2. **Definitions.** Let k be an algebraically closed field of characteristic zero, m and $(m-\alpha)$ two integers, d their g.c.d., so that $m=ad$ and $(m-\alpha)=bd$, where a and b are relatively prime.

Consider in the projective plane over k an irreducible curve of order m having a singular point P such that:

(1) P is the center of n places, all having the same tangent.

(2) For each $i=1, \cdots, n$ there exists an integer t_i such that a linear form has order at_i or bt_i at the ith place.

(3) $\sum t_i=d$ (so that $\sum at_i=m$, $\sum bt_i=m-\alpha$). Such a point will be called an $(m, m-\alpha)$-point. Such a curve will be called an $(m, m-\alpha)$-curve. A pencil of curves, depending linearly on a parameter λ, whose generic member has an $(m, m-\alpha)$-point that is independent of λ, and has its tangent independent of λ will be called an $(m, m-\alpha, \lambda)$-pencil.

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As the property of being an $(m, m-\alpha)$-curve is a local one we will use affine coordinates.

3. LEMMA A. *Given an $(m, m-\alpha, \lambda)$-pencil, a system of affine coordinates [in the projective plane] can be so chosen that the corresponding equation $F(x, y, \lambda)=0$ admits the following factorization*:

$$F(x, y, \lambda) = h \prod_{\epsilon\delta_\epsilon} (y - y_{\epsilon\delta_\epsilon})$$

where

h is a unit in Cl $k(\lambda)[y][[x^{1/bt}]]$,
the $y_{\epsilon\delta_\epsilon}$ are nonunits in Cl $k(\lambda)[[x^{1/bt}]]$,
ϵ ranges from 1 *to n*,
δ_ϵ ranges from 1 *to bt_ϵ*,
t is the l.c.m. of the t_i's,
Cl *denotes algebraic closure and the double bracket formal power-series ring.*

PROOF. It suffices to take as origin the $(m, m-\alpha)$-point, and $y=0$ as the tangent to the n corresponding places. The desired factorization is then obtained in the standard way, taking into account the existence of Puiseux-expansions at the origin and the form of the local Galois-group at that point.

Note that for the ith place there will be bt_i factors in this factorization, each of them of the form:

$$y - D_0(\theta^\eta x^{1/bt_i})^{at_i} - \sum_{v=1}^{\infty} D_v(\theta^\eta x^{1/bt_i})^{at_i+v}$$

where θ is a primitive bt_ith root of unity and $D_i \in$ Cl $k(\lambda)$.

4. LEMMA B. *Let k be any field, $yx_1 \cdots x_n$ and λ indeterminates. Suppose we have an identity*

$$g(yx_1 \cdots x_n) + \lambda f(yx \cdots x_n) = h \prod_{\epsilon=1}^{m} (y - y_\epsilon) \tag{4.1}$$

where

g and f are elements of $k[[x_1 \cdots x_n]][y]$,
the y_i are nonunits in Cl $k(\lambda)[[x_1 \cdots x_n]]$,
h is a unit in Cl $k(\lambda)[[yx_1 \cdots x_n]]$.

Order the terms of the power-series y_ϵ lexicographically in the $x_1 \cdots x_m$. In some of the y_ϵ certain coefficients may depend on λ. Suppose that the first such coefficient in every y_ϵ is separable over $k(\lambda)$. Then if two at least of the y_ϵ coincide in their first j terms the coefficients of these terms do not depend on λ.

PROOF. Suppose that in $\sigma \geqq 2$ of the m power series y_ϵ, say for $\epsilon=1, \cdots, \sigma$, the first term whose coefficient depends on λ is $a(\lambda)x_1^{\mu_1} \cdots x_n^{\mu_n}$, of total degree $r=\mu_1+\cdots+\mu_n$, and suppose that these series coincide up to this last term at least.

Define y^* as a polynomial in the $x_1 \cdots x_n$ such that

(1) $y^*=y_1$ (modulo terms of degree r),

(2) y^* has as terms of degree r the same terms as y_1 up to $a(\lambda)x_1^{\mu_1} \cdots x_n^{\mu_n}$, and instead of this term has the term $tx_1^{\mu_1} \cdots x_n^{\mu_n}$ where t is an indeterminate and has no higher terms. Note that y^* is linear in t. In Cl $k(\lambda)[[yx_1 \cdots x_nt]]$ we consider for (4.1) the specialization $y \to y^*$. We then get

$$g(y^*x_1 \cdots x_n) + \lambda f(y^*x_1 \cdots x_n) = h^* \prod_{\epsilon=1}^{m} (y^* - y_\epsilon) \tag{4.2}$$

where $g(y^*x_1 \cdots x_n)$ and $f(y^*x_1 \cdots x_n)$ are polynomials in t. Put

$$g(y^*x_1 \cdots x_n) = g^*(tx_1 \cdots x_n),$$
$$f(y^*x_1 \cdots x_n) = f^*(tx_1 \cdots x_n).$$

Obviously h^* remains a unit in Cl $k(\lambda)[[yx_1 \cdots x_nt]]$.

In (y^*-y_ϵ) the first term of every factor (y^*-y_ϵ) will have as coefficient

$$t - a(\lambda) \qquad \text{if } \epsilon = 1, 2, \cdots, \sigma,$$

and for any other ϵ it may be $t-c_\epsilon(\lambda)$ or an expression $d_\epsilon(\lambda)$ independent of t, or t. By hypothesis $a(\lambda)$, $c_\epsilon(\lambda)$, $d_\epsilon(\lambda)$ are separable over $k(\lambda)$. The first of the lowest degree terms in the right hand side of (4.2) will be obtained by multiplying the first terms in every factor—this gives a term of the form:

$$[t - a(\lambda)]^\sigma \phi(t, \lambda)x_1^{\xi_1} \cdots x_n^{\xi_n}$$

where ϕ is a polynomial in t whose coefficients are separable over $k(\lambda)$.

Therefore

$$g^*(tx_1 \cdots x_n) + \lambda f^*(tx_1 \cdots x_n) \equiv [t - a(\lambda)]^\sigma \phi x_1^{\xi_1} \cdots x_n^{\xi_n}$$

+different terms of same total degree+terms of higher degree.

Let $A(t)x_1^{\xi_1} \cdots x_n^{\xi_n}$ and $B(t)x_1^{\xi_1} \cdots x_n^{\xi_n}$ be the first of the lowest-degree terms in $g^*(tx_1 \cdots x_n)$ and $f^*(tx_1 \cdots x_n)$ respectively—A and B are polynomials in t.

Then

$$A(t) + \lambda B(t) = [t - a(\lambda)]^{\sigma}\phi(t, \lambda).$$

Differentiating with respect to λ we get

$$B(t) = [t - a(\lambda)]^{\sigma-1}\psi(t, \lambda) \quad (\psi \text{ polynomial in } t)$$

and $B(t)$ has $t=a(\lambda)$ as a root—a contradiction, since $B(t)$ is independent of λ.

5. MAIN THEOREM. *Consider an $(m, m-\alpha, \lambda)$-pencil in whose generic member the order of the divisor of singularities at its $(m, m-\alpha)$-point is equal to c. Then the following inequality holds:*

$$m^2 + \alpha + n \geqq \alpha a + 2m + c.$$

PROOF. By Lemma A we can choose a system of coordinates such that the corresponding equation of the $(m, m-\alpha, \lambda)$-pencil, $F(x, y, \lambda) = 0$, admits the factorization:

$$F = h \prod_{\epsilon,\delta_\epsilon} (y - y_{\epsilon\delta_\epsilon}). \tag{5.1}$$

It is well known (see [3]) that the order of the divisor of singularities at P for a curve $F=0$ is:

$$c = -\sum_{i=1}^{n} V_{P_i}\left[\frac{dx}{F_y}\right]$$

the sum ranging over the n places of $F=0$ at P. For the $(m, m-\alpha)$-curves we consider we then have:

$$\sum V_{P_i}[F_y] = c + \sum V_{P_i}[dx] = c + m - \alpha - n. \tag{5.2}$$

We will now compute an upper bound for $V_{P_i}[F_y]$ using (5.1).

We have

$$\sum V_{P_i}[F_y] = \sum V_{P_i}\left[\frac{\partial}{\partial y} h \prod_{\epsilon,\delta_\epsilon} (y - y_{\epsilon\delta_\epsilon})\right]. \tag{5.3}$$

At P_i we identify y with one of the roots $y_{i1}, \cdots, y_{ibt_i}$, say with y_{i1}. Then

$$\sum V_{P_i}[F_y] = \sum_i V_{P_i} \prod_{\epsilon\delta_\epsilon \neq i1} (y - y_{\epsilon\delta_\epsilon}). \tag{5.4}$$

We compute $V_{P_i}\prod_{\delta_i \neq 1} (y - y_{i\delta_i})$ first. Only for t_i of the δ_i will the first coefficient of $y_{i\delta_i}$ be equal to the first coefficient of y_{i1}, and in that case, by Lemma B, the two expansions $y_{i\delta_i}$ and y_{i1} coincide at most up to the first term whose coefficient depends on λ. Denote by

$\rho(\lambda)x(at_i+\beta_i)/bt_i$ the first term in y_{i1} whose coefficient depends on λ. Then

$$V_{P_i}\prod_{\delta_i\neq 1}(y-y_{i\delta_i}) \leqq (t_i-1)(at_i+\beta_i)+(bt_i-t_i)at_i.$$

For the remaining part, $V_{P_i}\prod_{j\neq 1}(y-y_{j\delta_j})$ a similar argument gives

$$V_{P_i}\prod_{j\neq 1}(y-y_{j\delta_j}) \leqq t_j(at_i+\beta_i)+(bt_j-t_j)at_i.$$

All in all,

$$\begin{aligned} &V_{P_i}\left[\frac{\partial}{\partial y}h\prod(y-y_{\epsilon\delta_\epsilon})\right] \\ &\qquad \leqq (at_i+\beta_i)\left(\sum t_j-1\right)+at_i\left[\sum_j(bt_j-t_j)\right]. \end{aligned} \tag{5.5}$$

Remembering that $\sum t_i=d$ we get

$$V_{P_i}[F_y] \leqq abdt_i-at_i+(d-1)\beta_i.$$

Taking the sum over all the places centered at P we have:

$$\sum_i V_{P_i}[F_y] \leqq abd^2+(d-1)\sum\beta_i-ad. \tag{5.6}$$

To evaluate $\sum\beta_i$ we consider two independent generic members of the pencil, with parameters λ, μ and the expression:

$$V_{P_i}(y_{i\delta_i}(\mu)-y_{\epsilon\delta_\epsilon}(\lambda)) \quad\text{with}\quad i\delta_i\neq\epsilon\delta_\epsilon, \tag{5.7}$$

the P_i's being taken on the λ-curve.

Using Lemma B we get in this case

$$V_{P_i}(y_{i\delta_i}(\mu)-y_{\epsilon\delta_\epsilon}(\lambda)) = V_{P_i}(y_{i\delta_i}(\lambda)-y_{\epsilon\delta_\epsilon}(\lambda)).$$

Therefore

$$\sum_i V_{P_i}\prod_{\epsilon\delta_\epsilon\neq i1}(y_{i1}(\mu)-y_{\epsilon\delta_\epsilon}(\lambda)) = \sum V_{P_i}[F_y]. \tag{5.8}$$

But the intersection-multiplicity of two generic members of the pencil at P is

$$\begin{aligned} &\sum_i V_{P_i}\prod(y_{i1}(\mu)-y_{\epsilon\delta_\epsilon}(\lambda)) \\ &\quad = \sum_i V_{P_i}\prod_{\epsilon\delta_\epsilon\neq i1}(y_{i1}(\mu)-y_{\epsilon\delta_\epsilon}(\lambda))+\sum_i V_{P_i}(y_{i1}(\mu)-y_{i1}(\lambda)). \end{aligned} \tag{5.9}$$

Moreover $\sum_i V_{P_i}(y_{i1}(\mu)-y_{i1}(\lambda)) = \sum_i(at_i+\beta_i) = m+\sum\beta_i$. This multiplicity is $\leqq m^2$, so that $\sum V_{P_i}[F_\nu]+m+\sum\beta_i\leqq m^2$. By (5.2) we have:

$$\sum \beta_i \leqq m^2 - 2m - c + \alpha + n. \tag{5.10}$$

From (5.2) and (5.6) follows:

$$c + m - \alpha - n \leqq abd^2 - ad + (d - 1)\sum \beta_i.$$

A fortiori,

$$c + m - \alpha - n \leqq abd^2 - ad + (d - 1)(m^2 - 2m - c + \alpha + n)$$

or finally

$$dm^2 + \alpha d + nd \geqq \alpha m + 2dm + cd. \tag{5.11}$$

6. PROPOSITION. *Let the generic curve of an $(m, m-\alpha, \lambda)$-pencil have as order of its divisor of singularities at its $(m, m-\alpha)$-point the number c, with $c>(m-1)(m-2)-2$. Then α divides m.*

Applying the main theorem,

$$m^2 + \alpha + n > \alpha a + 2m + (m - 1)(m - 2) - 2 = a\alpha + m^2 - m,$$

or

$$m + n > \alpha(a - 1).$$

Since $m=ad$ and $n\leqq d$ we have a fortiori

$$a + 1 > \frac{\alpha}{d}(a - 1).$$

As both $a>1$ and $d|\alpha$ this can only hold for $\alpha=d$.

COROLLARY A. *If the generic member of an $(m, m-\alpha, \lambda)$-pencil is a rational curve, and has no other singular point than its $(m-\alpha)$-point, then α divides m.*

PROOF. For the order of the divisor of singularities is $(m-1)(m-2)$ minus twice the genus.

7. JUNG'S THEOREM. *Consider the affine plane over an algebraically closed field k of characteristic 0. Then any automorphism of this plane is a product of linear transformations and transformations of the following type:*

$$x' = x,$$
$$y' = y + cx^n,$$

where $c\in k$ and n is a positive integer.

PROOF. Let σ be an automorphism of the plane given by

$$(7.1)\qquad \begin{aligned} x' &= f(x, y), \\ y' &= y(x, y), \end{aligned}$$

where f and g are polynomials, elements of $k[x, y]$ of degree n and m respectively, $n \geqq m$. For $n = m = 1$ there is nothing to prove. So we suppose $n \geqq 2$. As σ is everywhere biregular in the affine plane the jacobian of f and g is constant. It then follows that the highest degree terms of f and g, having jacobian zero, are, up to a constant factor, powers of a common polynomial h. In the projective plane σ defines a birational transformation which is well defined for a generic point of infinity. The same is true for σ^{-1}, so that to the point at infinity on a generic line of the affine plane there corresponds under σ^{-1} only one point at infinity on the corresponding curve. It follows that the polynomial h is a power of a linear form.

By a linear transformation (7.1) becomes:

$$(7.2)\qquad \begin{aligned} x' &= x^n + f_1(x, y), \\ y' &= x^m + g_1(x, y) \end{aligned}$$

where $n \geqq m$, $n \geqq 2$ and the degrees of f_1 and g_1 are respectively $\leqq n-1$, $\leqq m-1$. After factoring out a linear transformation if necessary we may assume $n > m$.

To the pencil $y = \lambda$, λ an indeterminate, there corrresponds under σ a pencil of rational curves. Consider their parametric representation, obtained from (7.2) by putting $y = \lambda$. It is readily seen that a generic member of this pencil has a unique point at infinity, P, independent of λ, center of only one place, with the line at infinity as tangent. Moreover a line through P intersects the curve in either n or $(n-m)$ points at P. Obviously a generic curve of the pencil has no singularities in the affine plane. Such a pencil is a particular case of a $(n, n-m, \lambda)$-pencil, and by Corollary A we get $m \mid n$.

Then $\sigma = w \circ \mu$, where w is the transformation

$$\begin{aligned} x' &= x'' + y''^{n/m}, \\ y' &= y'', \end{aligned}$$

and μ is of the type

$$\begin{aligned} x'' &= p(x, y), \\ y'' &= x^m + g_1(x, y), \end{aligned}$$

with p a polynomial of degree less than n. This proves the theorem.

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ON PROJECTIVE MODULES OVER SEMI-HEREDITARY RINGS

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This note contains a proof of the following

THEOREM. *Each projective module P over a (one-sided) semi-hereditary ring Λ is a direct sum of modules, each of which is isomorphic with a finitely generated ideal of* Λ.

This theorem, already known for finitely generated projective modules [**1**, I, Proposition 6.1], has been recently proved for arbitrary projective modules over commutative semi-hereditary rings by I. Kaplansky [**2**], who raised the problem of extending it to the noncommutative case.

We recall two results due to Kaplansky:

Any projective module (over an arbitrary ring) is a direct sum of countably generated modules [**2**, Theorem 1].

If any direct summand N of a countably generated module M is such that each element of N is contained in a finitely generated direct summand, then M is a direct sum of finitely generated modules [**2**, Lemma 1].

According to these results, it is sufficient to prove the following proposition:

Each element of the module P is contained in a finitely generated direct summand of P.

Let $F=P\oplus Q$ be a free module and x be an arbitrary element of P. Let $x=\lambda_1x_1+\cdots+\lambda_nx_n$ be a representation of the element x in some base for the free module F and let G denote the free submodule

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