TRANSFORMATION OF PROBABILITIES

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1. **Transformation.** Let $\boldsymbol{R}$ be a σ-algebra[2] of subsets of X, and $\mathcal{P}$ the set of all probability measures P on $\boldsymbol{R}$. Let T transform $\mathcal{P}$ into itself. For certain sets $E \in \boldsymbol{R}$, knowledge of P throughout E (i.e., for all subsets of E belonging to $\boldsymbol{R}$) determines TP throughout E. The class of sets having this property will be denoted by $\boldsymbol{E}_T$, or better, since T will be fixed, by $\boldsymbol{E}$. Evidently $\boldsymbol{E}$ contains 0, X, and the complements of atoms. We show that if $\boldsymbol{E}$ is sufficiently large, then T is a linear combination of the identity and a constant. There are applications to the theory of learning and to political theory [**1**; **3**; **4**; **6**].

THEOREM 1. (A) *If $\boldsymbol{E}$ contains an algebra $\boldsymbol{A}$ whose Borel extension is $\boldsymbol{R}$, and if $|\boldsymbol{R}| > 4$, then $TP \equiv \alpha P + (1-\alpha)P_0$, where $\alpha \leqq 1$ and $P_0 \in \mathcal{P}$.*

(B) *The converse is true with no restriction on $\boldsymbol{R}$.*

(C) *If $\boldsymbol{R}$ is infinite, then $\alpha \geqq 0$.*

In the political interpretation, the elements of X are parties (or political positions). P is the distribution of voters, T is the electoral mechanism, and TP the distribution of seats in the legislature. If T is the identity, the mechanism is Proportional Representation. If T is a constant, the political complexion of the legislature is fixed by law. It will be seen from Theorem 3 that $E \in \boldsymbol{E}$ means that if the complement $-E$ unites in a coalition, the effect is independent of whether this occurs before or after the election. $|\boldsymbol{R}| > 4$ means essentially that there are more than two parties. Part (A) of the theorem is not true for $|\boldsymbol{R}| = 4$.

In learning theory, P is a probability distribution of responses, and TP is a new distribution resulting from a learning experience. If T is the identity, there is no learning. If T is a constant, this is one-trial learning.

Bush, Mosteller, and Thompson [**4**] proved an equivalent theorem for the case $\boldsymbol{R}$ finite and $\boldsymbol{E} = \boldsymbol{R}$ (Corollary 3 of Theorem 3). Some of their ideas are used in the proof.

Denote by $\boldsymbol{B}$ the class of sets E such that $P(E) = Q(E)$ implies $TP(E) = TQ(E)$, for all P, $Q \in \mathcal{P}$. The importance of $\boldsymbol{B}$ is that for

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[2] Borel field. We follow the terminology of [**5**]. In addition, the Borel extension of a class $\boldsymbol{M}$ of sets is the smallest σ-algebra containing $\boldsymbol{M}$. This is the same as $S(\boldsymbol{M})$ if X is the countable union of sets in $\boldsymbol{M}$.

each proper set $E\in\boldsymbol{B}$, there is a function γ_E mapping $[0, 1]$ into itself such that $TP(E)=\gamma_E[P(E)]$. We have also $\gamma_0(0)=0$.

For any class $\boldsymbol{S}$ of subsets of X, let $\boldsymbol{S}^*$ denote the class of sets E for which the complement $-E\in\boldsymbol{S}$.

PROPOSITION 1. $\boldsymbol{E}\cap\boldsymbol{E}^*\subseteq\boldsymbol{B}$.

PROOF. Let E, $-E\in\boldsymbol{E}$. Let $P(E)=Q(E)$ for two members P, Q of $\mathcal{P}$. Define $P'\in\mathcal{P}$ as follows. For $A\in\boldsymbol{R}$, let $P'(A)=P(A-E)+A(c)P(E)$, where $c\in E$ and $A(x)$ is the characteristic function of A. Define Q' similarly. We have $P'\equiv Q'$ on E. Also $P\equiv P'$ and $Q\equiv Q'$ on $-E$. Hence $TP'\equiv TQ'$ on E, while $TP\equiv TP'$ and $TQ\equiv TQ'$ on $-E$. In particular, the last two equations are true for $-E$ itself, and, taking complements, also for E. We have $TP(E)=TP'(E)=TQ'(E)=TQ(E)$, proving $E\in\boldsymbol{B}$.

Since $\boldsymbol{A}$ is an algebra, $\boldsymbol{A}=\boldsymbol{A}^*$. Thus $\boldsymbol{B}\supseteq\boldsymbol{E}\cap\boldsymbol{E}^*\supseteq\boldsymbol{A}\cap\boldsymbol{A}^*=\boldsymbol{A}$, and so $\boldsymbol{B}\supseteq\boldsymbol{A}$.

Define the set function u on the class $\boldsymbol{A}-\{X\}$ as follows: $u(E)=\gamma_E(0)$. Using the fact that TP is a measure for each P, and choosing P so as to vanish on the appropriate sets, it is easy to show that u is a measure on the semiring $\boldsymbol{A}-\{X\}$, and therefore extends uniquely to a measure u on $\boldsymbol{R}$ [**5**; **7**]. Evidently $u\leqq 1$ on $\boldsymbol{A}-\{X\}$, but it would be incorrect to infer that $u(X)\leqq 1$.

PROPOSITION 2. *Let* $E\cap F=0$; E, F, $E\cup F$ *proper sets in* $\boldsymbol{B}$; x, y, $x+y\in[0, 1]$. *Then* $\gamma_{E\cup F}(x+y)=\gamma_E(x)+\gamma_F(y)$.

PROOF. Using all the hypotheses, it is easy to show that there is a probability measure P with $P(E)=x$ and $P(F)=y$. For this P,

$$\gamma_{E\cup F}(x + y) = TP(E \cup F) = TP(E) + TP(F) = \gamma_E(x) + \gamma_F(y).$$

If E and F are proper sets in $\boldsymbol{A}$, let $E\sim F$ denote the statement that $\gamma_E(x)-u(E)\equiv\gamma_F(x)-u(F)$ for all $x\in[0, 1]$. $\sim$ is an equivalence relation.

PROPOSITION 3. *The relation* $\sim$ *is universal on the proper sets in* $\boldsymbol{A}$.

PROOF. (1) Let $E\subset B$, where $\subset$ denotes proper inclusion. By Proposition 2, with $F=B-E$ and $y=0$, $\gamma_B(x)=\gamma_E(x)+\gamma_{B-E}(0)$. Letting $x=0$, $\gamma_B(0)=\gamma_E(0)+\gamma_{B-E}(0)$. Subtracting, we have $E\sim B$.

(2) If $E\cap F=0$ and $E\cup F\neq X$, then $E\sim E\cup F\sim F$ by (1).

(3) If E, F are incomparable and $E\cap F\neq 0$, then $E\sim E\cap F\sim F$ by (1).

(4) This leaves only the case $F=-E$. For the first time, we invoke the hypothesis $|\boldsymbol{R}|>4$, which easily implies $|\boldsymbol{A}|>4$. Hence E or F

must have a proper subset, say $E \supset A$. Then $E \sim A$ by (1) and $A \sim F$ by (2).

In view of Proposition 3 and $|\boldsymbol{A}| > 2$, the equation

$$\gamma(x) = \gamma_E(x) - u(E) \qquad \text{for } E \in \boldsymbol{A} - \{0, X\}$$

defines $\gamma(x)$ uniquely. γ maps $[0, 1]$ into $[-1, 1]$.

PROPOSITION 4. *$\gamma(x) \equiv \alpha x$, with $\alpha \leqq 1$.*

PROOF. Let $x, y, x+y \in [0, 1]$. Choose E, F so that E, F, $E \cup F$ are proper sets in $\boldsymbol{A}$, and so that $E \cap F = 0$. Here we have used $|\boldsymbol{R}| > 4$ for the second and last time. By Proposition 2 and the definitions of u and γ,

$$\begin{aligned} \gamma(x + y) + u(E \cup F) &= \gamma_{E \cup F}(x + y) = \gamma_E(x) + \gamma_F(y) \\ &= \gamma(x) + u(E) + \gamma(y) + u(F). \end{aligned}$$

Since u is additive, we conclude that $\gamma(x+y) = \gamma(x) + \gamma(y)$. A bounded function of this type is of the stated form. The proof in [**2**] can be adapted. Obviously, $\alpha \leqq 1$.

Thus $TP(E) = \alpha P(E) + u(E)$ for all E in the semiring $\boldsymbol{A} - \{X\}$. If $\alpha \geqq 0$, then $\alpha P + u$ is a measure on $\boldsymbol{R}$, equal to TP on $A - \{X\}$, and therefore on $\boldsymbol{R}$. If $\alpha \leqq 0$, then $TP - \alpha P$ is a measure on $\boldsymbol{R}$, equal to u on $\boldsymbol{A} - \{X\}$, and therefore on $\boldsymbol{R}$. In either case $TP = \alpha P + u$ on $\boldsymbol{R}$. In passing, note that

$$(1) \qquad 1 = TP(X) = \alpha + u(X).$$

If $\alpha = 1$, then $TP \equiv P$. If $\alpha < 1$, define P_0 by $(1-\alpha)P_0 = u$. The main assertion (A) of Theorem 1 follows.

Assertion (B) is immediate, taking $\boldsymbol{A} = \boldsymbol{R}$. For (C) we require a simple result from set theory. We omit the proof, which is not difficult.

LEMMA. *If $\boldsymbol{A}$ is an infinite algebra of subsets of X, then X is the union of a monotone sequence of sets of $\boldsymbol{A} - \{X\}$.*

To resume the proof of (C), the infinite cardinality of $\boldsymbol{R}$ implies the same for $\boldsymbol{A}$. Then we have $u(X) = \lim_{n \to \infty} u(E_n)$ for sets $E_n \in \boldsymbol{A} - \{X\}$. But $u \leqq 1$ on $\boldsymbol{A} - \{X\}$, and therefore $u(X) \leqq 1$. With (1), we have $\alpha \geqq 0$.

For applications to special cases, we need the following closure properties of $\boldsymbol{E}$, which are of independent interest.

THEOREM 2. (A) *If $E, F \in \boldsymbol{E}$, and $E \cup F \neq X$, then $E \cap F \in \boldsymbol{E}$.*

(B) *$\boldsymbol{E}$ is closed with respect to countable union.*

PROOF. (A) Let $P \equiv Q$ on $E \cap F$. Without loss of generality, assume $P(E) \leqq Q(E)$. Define a new probability measure P' as equal to P on E (i.e., throughout E), equal to Q on $F-E$, and arbitrary on $X-E-F$ except that $P'(E)+P'(F-E)+P'(X-E-F)=1$. For other sets, P' is defined by additivity. In verification that the values assigned on $X-E-F$ are feasible, we observe that this set is not empty and that $P'(E \cup F) \leqq Q(E \cup F) \leqq 1$.

Now $P \equiv P'$ on E and $Q \equiv P'$ on F. Hence $TP \equiv TP'$ on E and $TQ \equiv TP'$ on F. The last two identities are true, therefore, on $E \cap F$. Hence $TP \equiv TQ$ on $E \cap F$, and $E \cap F \in \boldsymbol{E}$.

We remark that when $E \cup F = X$, (A) is false in the strong sense that given such overlapping incomparable E, F, there exists a T for which E and F are in $\boldsymbol{E}_T$, but $E \cap F$ is not.

(B) Let $P \equiv Q$ on $E = \bigcup_1^\infty E_n$, where $E_n \in \boldsymbol{E}$. Then $P \equiv Q$ on E_n, which implies $TP \equiv TQ$ on E_n, for each n. Let $\{F_n\}$ be a disjoint sequence having the same partial unions as $\{E_n\}$. We have $TP \equiv TQ$ on F_n, since $F_n \subseteq E_n$. Then $TP \equiv TQ$ on E by countable additivity, and $E \in \boldsymbol{E}$.

The hypothesis of Theorem 1 may be expressed in two parts:

(I) $|\boldsymbol{R}| > 4$, *$\boldsymbol{E}$ contains a class $\boldsymbol{S}$ whose Borel extension is $\boldsymbol{R}$, and $X \in \boldsymbol{S}$.*

(II) *$\boldsymbol{S}$ is a ring.*

PROPOSITION 5. *In Theorem* 1, (II) *can be weakened to*: *$\boldsymbol{S}$ is a semiring.*

PROOF. The class of finite disjoint unions of elements of $\boldsymbol{S}$ is a ring [**5**]. Since it contains $\boldsymbol{S}$, this ring generates $\boldsymbol{R}$ and contains X. By Theorem 2B, $\boldsymbol{E}$ contains the ring.

EXAMPLES. In all of these, let $\boldsymbol{R}$ be the class of Borel sets.

(i) X = the real line. Let $\boldsymbol{E}$ contain all intervals $[\alpha, \beta)$. (Here and in the following it would suffice to take α and β rational.) Then E contains also $[\alpha, \infty)$ and $(-\infty, \alpha)$. With 0 and X, these finite and semi-infinite intervals constitute a semiring. Proposition 5 applies, and $TP \equiv \alpha P + (1-\alpha)P_0$ with $0 \leqq \alpha \leqq 1$ as in Theorem 1. This is equally true if $\boldsymbol{E}$ is assumed instead to contain all proper closed intervals.

(ii) (a) $X = (0, 1)$, (b) $X = [0, 1]$, (c) $X = [0, 1)$. Similar to Example (i).

(iii) X = Euclidean n-space ($n > 1$). Let $\boldsymbol{E}$ contain all half spaces $\{x: x_i \geqq \alpha\}$ and $\{x: x_i < \alpha\}$. Then $\boldsymbol{E}$ contains all finite intersections of these sets. (This implication is false for $n=1$.) With X added, these constitute a semiring, Proposition 5 applies, and T has the form stated in Theorem 1. This is true also if $\boldsymbol{E}$ is assumed to contain all slices $\{x: x_i \in [\alpha, \beta)\}$, or alternatively all cells

$$\{x: x_i \in [\alpha_i, \beta_i],\ i = 1, \cdots, n\}.$$

2. **Combination.** Bush and Mosteller [3] raised the question in learning theory of whether a set E could be shrunk to a point without making T ambiguous on the reduced space. More precisely, let $E \in \boldsymbol{R}$. A transformation C of $\mathcal{P}$ into itself is called a *combination* of E if it satisfies

$$\text{(C1)} \qquad CP \equiv P \text{ on } -E$$

and

$$\text{(C2)} \qquad P(E) = Q(E) \text{ implies } CP \equiv CQ \text{ on } E.$$

For example, let $c \in E$, and let

$$\text{(2)} \qquad CP(A) = P(E - A) + A(c)P(E) \qquad \text{for each } A \in \boldsymbol{R}.$$

We say that $E \in \mathbf{C}$ (E is combinable) if for each combination C of E, and for each $P \in \mathcal{P}$, we have

$$\text{(3)} \qquad CTCP = CTP.$$

In learning theory, (3) is called the *Combining* of *Classes* condition.

THEOREM 3. $\mathbf{C} = \boldsymbol{E}^*$.

PROOF. Let $E \in \mathbf{C}$, and C be a combination of E. We observe first that (C1) and (C2) imply

$$\text{(C3)} \qquad CP = CQ \text{ if and only if } P \equiv Q \text{ on } -E.$$

Now let $P \equiv Q$ on $-E$. Then $CP = CQ$. Hence $CTP = CTCP = CTCQ = CTQ$. Then a second use of (C3) yields $TP \equiv TQ$ on $-E$. This proves $\mathbf{C} \subseteq \boldsymbol{E}^*$.

Let $-E \in \boldsymbol{E}$. Let C combine E. Then $P \equiv CP$ on $-E$, and therefore $TP \equiv TCP$ on $-E$. By (C3), $CTP = CTCP$. Thus $\boldsymbol{E}^* \subseteq \mathbf{C}$.

COROLLARY 1. *In Theorem* 1, *the hypothesis that* $\boldsymbol{E}$ *contains the algebra* $\boldsymbol{A}$ *can be replaced by* $\mathbf{C} \supseteq \boldsymbol{A}$.

COROLLARY 2. (A) *The union of two overlapping sets of* $\mathbf{C}$ *is in* $\mathbf{C}$.
(B) $\mathbf{C}$ *is closed with respect to countable intersection.*

COROLLARY 3. *Let* X *be finite,* $|X| > 2$, *let* $\boldsymbol{R}$ *be the class of all subsets of* X, *and* $\mathbf{C} = \boldsymbol{R}$. *Then* $TP \equiv \alpha P + (1-\alpha)P_0$.

This is the Bush-Mosteller-Thompson theorem [4] mentioned earlier. Bush and Mosteller [3] showed that $\alpha(|X|-1)\geqq -1$. This bound is attained.

Regarding Example (ii)(c) as the real numbers modulo 1, let $\boldsymbol{C}$ contain all intervals $[\alpha, \beta)$, naturally including the case $\alpha<1<\beta$. With 0, this class is a semiring $\boldsymbol{S}$. Since $\boldsymbol{S}=\boldsymbol{S}^*$, also $\boldsymbol{E}\supseteq\boldsymbol{S}$, so that Proposition 5 applies, and T has the familiar form of Theorem 1. In Example (i) (the real line) the corresponding implication is false, even with the additional assumption that $\boldsymbol{C}$ contains all semi-infinite intervals, and similarly for Example (ii). To prove this, we use

PROPOSITION 6. *Let T be a combination of E of type* (2). *Then*

$$\boldsymbol{C}_T = [\{\{c\}\}^+ \cup \{E\}^- \cup \{-E\}^-] \cap \boldsymbol{R}.$$

(For any subset S of X, $\{S\}^-$ and $\{S\}^+$ denote respectively the class of subsets of S and the class of supersets of S.) The proof is omitted.

Returning to Example (i), let T be that combination of $E=[c, \infty)$ of type (2) which concentrates $P(E)$ at c. We see that $\boldsymbol{C}$ contains all the finite and infinite intervals mentioned above, but that intervals $[\alpha, \beta)$ containing c are not in $\boldsymbol{E}$, and the conclusion of Theorem 1 is false here.

3. **Partition.** A related problem, motivated by learning theory and political theory, is the following. For $n>1$, let $\mathcal{P}_n$ denote the class of all partitions of X into exactly n nonempty parts X_i, and let $\mathcal{C}_n$ denote the subclass of partitions (called combinable) for which the n-tuple $[P(X_1), \cdots, P(X_n)]$ uniquely determines $[TP(X_1), \cdots, TP(X_n)]$. It is not difficult to show that if each $X_i\in\boldsymbol{C}$, then $(X_1, \cdots, X_n)\in\mathcal{C}_n$. The converse is false, so that the latter statement is actually weaker than the former. Despite this, we have

THEOREM 4. *If $\mathcal{C}_n=\mathcal{P}_n$ for some $n<\log_2|\boldsymbol{R}|$, then $\boldsymbol{E}=\boldsymbol{C}=\boldsymbol{R}$, and $TP\equiv\alpha P+(1-\alpha)P_0$.*

PROOF. First we show that $E\in\boldsymbol{E}$ for all E divisible into $n-1$ (proper) parts. We can assume $E\neq X$. Let $P\equiv Q$ on E, and let $A\subseteq E$. We can express A as $\bigcup_1^a A_i$, where either $a=n-1$ or each A_i is atomic. If $a<n-1$, then $E-A=\bigcup_{a+1}^{n-1} A_i$ by the hypothesis on E. In either case, $P(A_i)=Q(A_i)$ for $i=1, \cdots, n-1$ and $P(-E)=Q(-E)$. Hence $TP(A_i)=TQ(A_i)$. Summing from 1 to a, $TP(A)=TQ(A)$, and $E\in\boldsymbol{E}$.

Evidently $E\in\boldsymbol{E}$ is proved unless E consists of the union of fewer than $n-1$ atoms. Let E be the union of $n-2$ atoms. Since $|\boldsymbol{R}|>2^n$, $-E$ has three parts, A, B, C. Then $E\cup A$ and $E\cup B$ are in $\boldsymbol{E}$, their union is not X, and so their intersection E is in $\boldsymbol{E}$ by Theorem 2A. Similarly for $n-3$, etc. Thus $\boldsymbol{E}=\boldsymbol{R}$, and the remaining statements follow from Theorems 1 and 3.

The theorem is false for $n \geqq \log_2 |\boldsymbol{R}|$.

Let $\mathcal{I}_n$ denote the class of partitions of X into n nonempty parts X_i, each of which is in a fixed semiring $\boldsymbol{S}$ whose Borel extension is $\boldsymbol{R}$. (The notation is suggested by examples where $\boldsymbol{S}$ consists of intervals.)

When X is the real line, and $\boldsymbol{S}$ the class of intervals $[\alpha, \beta)$, $(-\infty, \alpha)$, $[\alpha, \infty)$, the example at the end of §2 shows that $0 \neq \mathcal{I}_n \subseteq \mathcal{C}_n$ for all n does not imply $\boldsymbol{E}=\boldsymbol{R}$. The same is true for X a finite interval. The situation is different for a circle.

THEOREM 5. *Let X be the set of real numbers modulo 1, and $\boldsymbol{S}$ the class of intervals $[\alpha, \beta)$. If $\mathcal{I}_n \subseteq \mathcal{C}_n$ for some n, then $\boldsymbol{E}=\boldsymbol{C}=\boldsymbol{R}$, and $TP \equiv \alpha P+(1-\alpha)P_0$.*

PROOF. If $n=2$, then evidently $\boldsymbol{S} \subseteq \boldsymbol{B}$. With the single exception of Proposition 3, the proof of Theorem 1A applies, with the semiring $\boldsymbol{S}$ replacing the algebra $\boldsymbol{A}$. Proposition 1 is superfluous. We show now that the conclusion of Proposition 3 holds also in the present context. All intervals mentioned are proper, i.e., not 0 or X.

(1) Let $I_1 \subset I$. If $I-I_1$ is an interval, then $I_1 \sim I$ as in Proposition 3. If $I-I_1$ is not an interval, then it is the union of two disjoint intervals I_2 and I_3. Moreover, $I_1 \cup I_2$ is an interval. Thus $I_1 \sim I_1 \cup I_2 \sim I$.

(2) If $I_1 \cap I_2=0$ and $I_1 \cup I_2 \neq X$, then there is a proper interval I containing $I_1 \cup I_2$. Then $I_1 \sim I \sim I_2$ by (1).

Thus (1) and (2) in the proof of Proposition 3 are true in our present case. (3) and (4) apply unchanged. (This proof that $\boldsymbol{S} \subseteq \boldsymbol{B}$ implies the linearity of T is valid also for $X=$ the real line with $\boldsymbol{S}$ all $[\alpha, \beta)$, and for $X=$ Euclidean n-space with $\boldsymbol{S}$ all semiclosed cells.) This completes the proof for $n=2$.

Next, let $n>2$. Note that $P \equiv Q$ on I if and only if $P \equiv Q$ for all subintervals of I touching an end point. Hence $I \in \boldsymbol{E}$ if and only if the equality of P and Q for all such subintervals implies the same for TP and TQ.

Let $P \equiv Q$ on I, and let I_1 be a subinterval touching an end point. Write the interval $I-I_1$ as the disjoint union $\bigcup_3^n I_j$, and let $I_2=-I_1$. (Here we have used the fact that $\boldsymbol{S}=\boldsymbol{S}^*$.) We have $(I_1, \cdots, I_n) \in \mathcal{C}_n$, and $P(I_j)=Q(I_j)$ for all j. Hence $TP(I_j)=TQ(I_j)$ for all j, and in particular for $j=1$. Since I_1 was arbitrary, this proves $I \in \boldsymbol{E}$. Thus $\boldsymbol{S} \subseteq \boldsymbol{E}$.

Using $\boldsymbol{S}=\boldsymbol{S}^*$ again, we have $\boldsymbol{S} \subseteq \boldsymbol{E} \cap \boldsymbol{E}^*$. By Proposition 1, $\boldsymbol{S} \subseteq \boldsymbol{B}$, and the first part of the proof applies.

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