REASONABLE TOPOLOGIES FOR HOMEOMORPHISM[1] GROUPS

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1. **Introduction.** Let X be a homogeneous space and G a transitive group of homeomorphisms of X. We denote the isotropy group at x by G_x. If G is topologized in such a way that it is a topological group and the function $\phi: G/G_x \rightarrow X$ defined by $\phi(gG_x) = g(x)$ is a homeomorphism, then G is said to be *reasonably* topologized [**3**]. It is not difficult to see that if G is reasonable over X, it is admissibly topologized, i.e., the function $G \times X \rightarrow X$ defined by $(g, y) \rightarrow g(y)$ is continuous. In [**3**, Theorem 4.1], Ford gives a sufficient condition for the group of all homeomorphisms to be reasonably topologized over a completely regular space. However, his proof actually proves something less than that stated (see Proposition 3.1 below), and in fact, there is considerable doubt that the theorem as stated can be true, although the author knows no counter example. In this note, we give several conditions which will insure the result or a slightly weaker result. Several questions arise in the course of this study, and these are mentioned in §4.

2. **Some preliminaries.**

If G is a group of homeomorphisms of a uniform space X, we can induce a uniform space on G as follows: for each $A \in \mathfrak{A}$, the uniform structure for X, define $\hat{A} = \{(g, h): g, h \in G \text{ and } (g(x), h(x)) \in A, x \in X\}$. It is easy to see that $\hat{\mathfrak{A}} = \{\hat{A}: A \in \mathfrak{A}\}$ forms a uniform structure for G. However, G need not be a topological group under this structure. The following result is proved in [**3**]:

PROPOSITION 2.1 (FORD). *If G is a group of uniformly continuous homeomorphisms on a uniform space X, then G is a topological transformation group on X when given the uniform topology.*

PROPOSITION 2.2. *Let G be a group of homeomorphisms of a uniform space X, and suppose G is given the induced uniform structure. Then the function*

$$\phi_x: G \rightarrow X$$

and the function

$$\phi_x^*: G/G_x \rightarrow X$$

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defined by

$$\phi_x(g) = \phi_x^*(gG_x) = g(x)$$

are continuous, where G/G_x *is given the identification topology. Moreover,* ϕ_x^* *is one-one.*

PROOF. Let $\mathfrak{A}$ be the uniform structure on X, and $\hat{\mathfrak{A}}$ the induced uniform structure on G. If $A \in \mathfrak{A}$, and $\hat{A}$ the corresponding element of $\hat{\mathfrak{A}}$, then $\hat{A}(k)(x) \subset A(k(x))$, $k \in G$. Hence, ϕ_x is continuous at k for every $k \in G$.

Now if U is open in X, $\phi_x^{*-1}(U)$ is open if and only if $\pi^{-1}\phi_x^{*-1}(U)$ is open in G where $\pi: G \to G/G_x$ is the natural projection (which, incidentally, is not necessarily open). But

$$\pi^{-1}\phi_x^{*-1}(U) = \phi_x^{-1}(U)$$

which is open since ϕ_x is continuous. Clearly, ϕ_x^* is one-one.

DEFINITION 2.2. If G is a group which is also a topological space but not necessarily a topological group, and ϕ_x^* of Proposition 2.2 is a homeomorphism then we shall say that G is *quasi-reasonable* over X. If G is quasi-reasonable and a topological group, it is *reasonable.*

REMARK. Although this definition of reasonable differs slightly from Ford's, the two definitions are equivalent by the remark in the introduction.

PROPOSITION 2.3. *If G is a transitive group of homeomorphisms of X topologized so that left or right multiplication in G is continuous, then if* $\phi_x: G \to X$ *defined by* $\phi_x(g) = g(x)$ *is continuous (open) at the identity, it is continuous (open) everywhere.*

PROOF. Since multiplication on the left is continuous, by the existence of inverses, multiplication on the left is a homeomorphism. The result then follows quite trivially.

3. **Reasonable topologies.** The following notion was introduced by Ford [3].

DEFINITION 3.1. A space X is *strongly locally homogeneous* (SLH) if for any $x \in X$, and neighborhood V of x, there is a neighborhood U of x, $U \subset V$, such that for any $y \in V$, there is a homeomorphism g of X such that $g(x) = y$, and $g(z) = z$ if $z \notin U$. A homeomorphism with this property will be called an *(SLH)-homeomorphism.*

The following result is what Ford actually proved in his Theorem 4.1.

PROPOSITION 3.1. *If X is a completely regular, homogeneous, (SLH) space, then the function* $\phi_x: G \to X$ *defined by* $\phi_x(g) = g(x)$ *is open at the* *identity under any topology for G induced by a uniform structure for X, where G is any group containing a transitive subgroup generated by (SLH) homeomorphisms.*

THEOREM 3.1. *If X is a homogeneous, locally compact Hausdorff (SLH) space, then the uniform structure on X obtained from its one-point compactification induces on the group G of all homeomorphisms a reasonable topology over X.*

PROOF. Let $\hat{X}$ be the one-point compactification for X. For each $g \in G$, extend g to X by $\hat{g}(\infty)=\infty$, $\hat{g}(x)=g(x)$ if $x \in X$. Let $\hat{\mathfrak{A}}$ be a uniform structure for X. Then each $\hat{g}$ is uniformly continuous since X is compact. If $\mathfrak{A}$ is $\hat{\mathfrak{A}}$ cut down to X, then g is uniformly continuous over X relative to $\mathfrak{A}$. Hence, by Proposition 2.1, G is a topological group with the induced uniformity. By Propositions 2.3 and 3.1, the result follows.

This topology for G is precisely the g-topology of Arens [1] and is thus the coarsest topology under which G is reasonable over X.

DEFINITION 3.2. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be uniformities for the topological space X. We shall say that $\mathfrak{A}_1$ is *finer* than $\mathfrak{A}_2$ (or $\mathfrak{A}_2$ is *coarser* than $\mathfrak{A}_1$) if for any $B \in \mathfrak{A}_2$, there is an $A \in \mathfrak{A}_1$ with $A \subset B$. (Note that the topology for X is unaffected by this definition.)

The following result is well known:

PROPOSITION 3.2. *Let X be a locally compact Hausdorff space. The coarsest uniformity for X is that obtained from the (essentially unique) uniformity on its one-point compactification.*

COROLLARY. *Let G be a group of homeomorphisms of a locally compact Hausdorff space X. Any topology for G which is induced from a uniformity on X is finer than the g-topology, and thus G is admissible* [1] *under any such topology.* (*G may not be a topological group, however.*)

PROOF. This follows immediately from the definitions, Proposition 3.2, and the comment after Theorem 3.1.

THEOREM 3.2. *If X is a homogeneous, completely regular Hausdorff (SLH) space, then the uniform structure on X obtained from its Stone-Čech compactification induces on the group G of all homeomorphisms of X a reasonable topology over X.*

PROOF. Let $\mathfrak{B}(X)$ be the Stone-Čech compactification of X. Then each element g of G, considered as a function from X to $\mathfrak{B}(X)$, since it is continuously extendable to $\mathfrak{B}(X)$, is uniformly continuous in the

Stone-Čech uniformity. By Proposition 2.1, G is a topological group with the topology of uniform convergence, and by Proposition 2.3 and 3.1, G is reasonable over X.

DEFINITION 3.3. A uniformity $\mathfrak{A}$ of a space X is said to be *totally bounded* if its completion is compact.

The following result is well known and may be found in [**4**].

LEMMA 3.1. *The finest totally bounded uniform structure on a space X is its Stone-Čech uniformity.*

THEOREM 3.3. *If X is a homogeneous, completely regular Hausdorff (SLH) space, then any totally bounded uniform structure induces in the group G of all homeomorphisms a quasi-reasonable topology.*

PROOF. Let G denote the group of all homeomorphisms under uniform convergence relative to the given uniformity, and G' the same group under uniform convergence in the Stone-Čech uniformity. Since the latter uniformity is finer than the former, the topology induced on G' is finer than the topology induced on G, and hence, the identity function $i: G' \to G$ is continuous. Consider the following diagram:

$$\overset{\phi_x'}{\overbrace{G' \underset{i}{\to} G \underset{\phi_x}{\to} X}}$$

Since ϕ_x' is open, i is continuous, and $\phi_x' = i\phi_x$, ϕ_x is open. By Proposition 2.2, ϕ_x is continuous, and these two imply $\phi_x^*: G/G_x \to X$ is a homeomorphism.

THEOREM 3.4. *If X is a locally compact connected (SLH) Hausdorff space, then for any uniformity $\mathfrak{A}$ on X, the group of homeomorphisms uniformly continuous relative to $\mathfrak{A}$ is reasonable over X.*[2]

PROOF. Since X is (SLH) and arcwise connected, the group generated by "small" (SLH) homeomorphism is transitive over X and consists of uniformly continuous homeomorphisms. By Propositions 2.1 and 3.1, the theorem follows.

4. Final comments. It is unknown to the author whether or not ϕ_x in Proposition 3.1 is actually open throughout G, although it looks unlikely. However, something weaker than left (or right) continuity seems to be sufficient. There are several interesting problems that

[2] In general, if g is a uniformly continuous homeomorphism, g^{-1} need not be. By this we mean the group of all homeomorphisms g such that g and g^{-1} are both uniformly continuous.

arise in this connection. For example, in Theorem 3.3, can *quasi-reasonable* be replaced by reasonable if X is locally compact? If G is quasi-reasonable over X, is there a topology for G under which G is reasonable? Is there a weaker form of uniform continuity that will make ϕ_x open in Proposition 3.1, or will make multiplication in G continuous when G is topologized by this weaker notion of convergence?

The class of (SLH) spaces is known to contain the manifolds, the zero-dimensional completely regular spaces [3], and the universal curve [2]. Are there other examples of interest? If X is compact, and G a transitive group of homeomorphisms of X, can G be given a reasonable topology?

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