

ON ORTHOGONAL POLYNOMIALS¹

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This note is concerned with the following question: can any n distinct real numbers be the zeroes of a real orthogonal polynomial of degree n ? The answer is yes, and in fact even more is true as the following shows.

THEOREM. *Let $a < x_1 < x_2 < \cdots < x_n < b$, and $x_i < y_i < x_{i+1}$, $i = 1, 2, \cdots, n-1$. Then there exists an infinite set of polynomials $P_k(x)$, $k = 0, 1, \cdots$, where the degree of P_k is k , which are orthogonal on $[a, b]$, such that $P_n(x) = (x - x_1) \cdots (x - x_n)$ and $P_{n-1}(x) = (x - y_1) \cdots (x - y_{n-1})$.*

PROOF. We shall construct a set of polynomials $P_k(x)$, $k = 0, 1, \cdots$, with $P_0 \equiv 1$, P_n and P_{n-1} as given in the statement of the theorem, such that

$$(1) \quad P_k(x) = (x - a_k)P_{k-1}(x) - \lambda_k P_{k-2}(x), \quad k = 1, 2, \cdots,$$

where a_k is real, $\lambda_k > 0$, and $P_{-1} \equiv 0$, and such that all the zeroes of all the P_k are in the interior of $[a, b]$. The $P_k(x)$ will be the desired orthogonal polynomials.

First, let $a_n = x_1 + \cdots + x_n - y_1 - \cdots - y_{n-1}$. Then $P_n(x) - (x - a_n) \cdot P_{n-1}(x)$ is a polynomial of degree at most $n-2$, which we write as $-\lambda_n R(x)$, where $R(x) = (x - z_1) \cdots (x - z_r)$, $r \leq n-2$. Now $0 = P_n(x_1) = (x_1 - a_n)P_{n-1}(x_1) - \lambda_n R(x_1)$; however, $x_1 - a_n = (y_1 - x_2) + \cdots + (y_{n-1} - x_n) < 0$, and $P_{n-1}(x_1) \neq 0$, therefore, $\lambda_n \neq 0$ and $R(x_1) \neq 0$. Furthermore, $P_n(y_i) = -\lambda_n R(y_i)$, and since $P_n(y_i)$ and $P_n(y_{i+1})$ have opposite sign, it must be that $r = n-2$ and that after reordering, $y_i < z_i < y_{i+1}$, $i = 1, 2, \cdots, n-2$. Let $P_{n-2}(x) = R(x)$. Finally, $\lambda_n = (x_1 - a_n)P_{n-1}(x_1)/P_{n-2}(x_1) > 0$ since $P_{n-1}(x_1)$ and $P_{n-2}(x_1)$ have opposite sign. We can now repeat this procedure to obtain $P_{n-3}, P_{n-4}, \cdots, P_1, P_0 \equiv 1$. Thus, $P_k(x)$, $k = 0, 1, \cdots, n$ are defined and satisfy (1). To obtain $P_k(x)$ for $k > n$ set

$$P_{n+j}(x) = (x - a_{n+j})P_{n+j-1}(x) - \lambda_{n+j}P_{n+j-2}(x), \quad j = 1, 2, \cdots,$$

where a_{n+j} and $\lambda_{n+j} > 0$ are chosen so that the zeroes of P_{n+j} are in (a, b) .

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Having constructed the $P_k(x)$ to satisfy (1) we now apply a result of Favard [1] who showed that the P_k are orthogonal on some interval with respect to some distribution, and a result of Sherman [4] to the effect that the interval of orthogonality of a set of orthogonal polynomials may be taken to be any interval which contains all the zeroes of all the polynomials in its interior.

Even though this completes the proof of the theorem, we would like to give a short proof that if a and b are both finite, then the $P_k(x)$ are orthogonal on $[a, b]$, with respect to some distribution. We have only to show that there exists a nondecreasing function $\sigma(x)$ such that

$$(2) \quad \int_a^b d\sigma = 1, \quad \int_a^b P_k(x) d\sigma = 0, \quad k > 0,$$

for then it follows from (1) that

$$\int_a^b P_k(x) x^l d\sigma = 0, \quad 0 \leq l < k.$$

Now, (2) is a moment problem of a general type considered by Krein [2]. His solution is that the moment problem $\int_a^b P_k(x) d\sigma = c_k$ has a solution if and only if whenever $\sum_{k=0}^n \alpha_k P_k(x) \geq 0$ on $[a, b]$ then $\sum_{k=0}^n \alpha_k c_k \geq 0$, for all n . Therefore, suppose $\sum_{k=0}^n \alpha_k P_k(x) \geq 0$ on $[a, b]$. Since $\sum_{k=1}^n \alpha_k P_k(x)$ vanishes at least once in (a, b) (see [3]), $\alpha_0 \geq 0$, and therefore $\sum_{k=0}^n \alpha_k c_k = \alpha_0 \geq 0$.

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