HOLONOMY, RICCI TENSOR AND KILLING VECTOR FIELDS[1]

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1. Let M be a Riemannian manifold, connected and of class C^∞. For any vector field X on M, we define a tensor field A_X of type (1, 1), namely, a field of linear endomorphisms of the tangent space at each point, by setting $A_X \cdot Y = -\nabla_Y X$, where Y is a tangent vector at an arbitrary point and ∇_Y denotes covariant derivative with respect to Y.

It is known [2] that if X is a Killing vector field, then A_X is a skew-symmetric endomorphism of the tangent space and belongs to the normalizer of the holonomy algebra (Lie algebra of the homogeneous holonomy group) at each point of M. A result of Lichnerowicz [3] implies that if the restricted homogeneous holonomy group is irreducible and if the Ricci tensor is not zero, then A_X belongs to the holonomy algebra at each point. One of the basic contributions in contemporary Riemannian geometry is the result, due to Kostant [2], that the same conclusion holds if M is compact. His proof uses the Green-Stokes formula which is valid only on a compact Riemannian manifold.

In the present note, we wish to provide a more geometric proof to this theorem of Kostant, in fact, in a generalized form where the compactness of the space is not assumed. Namely, we shall prove

THEOREM. *Let M be a complete Riemannian manifold. If X is a Killing vector field defined on M which attains a local maximum in its length at some point of M, then A_X belongs to the holonomy algebra at each point of M.*

Here we say that the length $\|X\|$ of a vector field X attains a local maximum at a point $x \in M$ if x has a neighborhood U such that $\|X\|_y \leqq \|X\|_x$ for every point $y \in U$. The assumption of our theorem is valid, for example, if X has constant length on M, or if M is compact. The theorem of Kostant follows immediately.

2. We now sketch the proof of our theorem. First, we may assume that M is simply connected. Otherwise, let $\hat{M}$ be the universal covering manifold of M provided with a natural Riemannian metric so

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that the projection π of $\hat{M}$ onto M is a local isometry. $\hat{M}$ is complete. The holonomy algebra at $\hat{x}\in\hat{M}$ is the same as the holonomy algebra at $x=\pi(\hat{x})$. The given Killing vector field X on M can be lifted to a Killing vector field $\hat{X}$ on $\hat{M}$ which projects upon X. Suppose that $\|X\|$ attains a local maximum at $x\in M$. It is then clear that $\|\hat{X}\|$ attains a local maximum at any point $\hat{x}\in\hat{M}$ such that $\pi(\hat{x})=x$. If the conclusion of the theorem holds for $A_{\hat{X}}$, then it holds obviously for A_X.

Thus, let us assume that M is simply connected and complete. Let $M=M_0\times M_1\times\cdots\times M_k$ be the de Rham decomposition of M, where M_0 is a euclidean space and each M_i, $1\leqq i\leqq k$, is irreducible [4]. The given Killing vector field X on M decomposes into a sum $X_0+X_1+\cdots+X_k$, where each X_i, $0\leqq i\leqq k$, is a Killing vector field on M_i. Let $x=(x_0, x_1, \cdots, x_k)\in M_0\times M_1\times\cdots\times M_k$ be a point where $\|X\|$ attains a local maximum. Since $\|X\|_x^2=\|X_0\|_{x_0}^2+\|X_1\|_{x_1}^2+\cdots+\|X_k\|_{x_k}^2$, each vector field X_i on M_i attains a local maximum at the point x_i for the following reason. Suppose that this is not the case and that for some i, $x_i\in M_i$ has an arbitrarily nearby point $y_i\in M_i$ such that $\|X_i\|_{y_i}>\|X_i\|_{x_i}$. Then we can get a point $y=(x_0, x_1, \cdots, y_i, \cdots, x_k)$ at which $\|X\|_y>\|X\|_x$ and which is arbitrarily near x, contrary to the assumption that $\|X\|$ attains a local maximum at x. It is clear that we have only to prove the theorem for each X_i on M_i.

We now consider each vector field X_i on M_i. On the euclidean space M_0, the length of a Killing vector field X_0 cannot have a local maximum unless it is constant, in which case, X_0 is a parallel vector field and the corresponding endomorphism A_{X_0} is zero at every point. On each M_i, $1\leqq i\leqq k$, we make the following argument. The holonomy algebra of M_i is irreducible. If the Ricci tensor of M_i is not identically zero, then A_{X_i} belongs to the holonomy algebra by the result of Lichnerowicz as we already mentioned. To deal with the factor M_i whose Ricci tensor is identically zero, we need the following two general lemmas whose proofs will be given in §4.

LEMMA 1. *Let X be a Killing vector field on a Riemannian manifold. Then*

$$\operatorname{div}(A_X\cdot X) = -S(X, X) - \operatorname{trace}(A_X^2),$$

where $S(X, X)$ is the quadratic form in X given by the Ricci tensor.

LEMMA 2. *For a Killing vector field X on a Riemannian manifold with Riemannian metric g, let $\phi=(1/2)\|X\|^2$. For any vector field V with $\nabla_V V=0$ in a neighborhood of a point x, we have*

$$V^2\phi = g(V, \nabla_V(A_X \cdot X)).$$

Now assume that the Ricci tensor S is identically zero. Lemma 1 gives $\operatorname{div}(A_X \cdot X) = -\operatorname{trace}(A_X)^2$. If $\|X\|$ attains a local maximum at x, we have $V^2\phi = g(V, \nabla_V(A_X \cdot X)) \leqq 0$ at x in Lemma 2. Since $\operatorname{div}(A_X \cdot X)$ is the trace of the linear mapping $V \to \nabla_V(A_X \cdot X)$ of the tangent space at x into itself, we see that $\operatorname{div}(A_X \cdot X) \leqq 0$ at x. On the other hand, A_X being skew-symmetric, we have $\operatorname{trace}(A_X)^2 \leqq 0$ at x. Therefore we must have $\operatorname{div}(A_X \cdot X) = \operatorname{trace}(A_X)^2 = 0$ at x, which is possible only when $A_X = 0$ at x. Thus A_X belongs, of course, to the holonomy algebra at x. As was shown in [2], it follows that A_X belongs to the holonomy algebra at each point of M. This concludes the proof of our theorem.

3. A similar argument allows us to prove a theorem of Bochner [1] in the following form.

THEOREM. *Let M be a Riemannian manifold whose Ricci tensor is negative definite. If a Killing vector field X attains a local maximum in its length at some point of M, then X is identically zero.*

In fact, by the same argument following the above lemmas, we have $-S(X, X) - \operatorname{trace}(A_X)^2 \leqq 0$ at x. On the other hand, since S is negative definite, we must have $-S(X, X) \geqq 0$ everywhere. We have also $-\operatorname{trace}(A_X)^2 \geqq 0$. Thus, at x, we have $S(X, X) = 0$ and $\operatorname{trace}(A_X)^2 = 0$, which imply that $X = 0$ and $A_X = 0$ at x. By a well known fact that a Killing vector field on a connected Riemannian manifold is uniquely determined by the values of X and A_X at an arbitrary single point [2], we see that X is zero on the whole manifold.

4. For the sake of completeness, we shall give here proofs of Lemmas 1 and 2.[2]

PROOF OF LEMMA 1. The Ricci tensor is given, by definition, by $S(X, Y) =$ trace of the linear endomorphism $V \to R(V, X)Y$ of the tangent space at each point, where R is the curvature tensor and $R(V, Y)$ is the skew-symmetric endomorphism obtained by contraction of R with vectors V and Y. Now assume that X is a Killing vector field and Y is an arbitrary vector field. We have $\nabla_V(A_X) = R(X, V)$ (see [2]), and hence $-R(X, V)Y = -(\nabla_V(A_X))Y = -\nabla_V(A_X \cdot Y) + A_X(\nabla_V Y) = -\nabla_V(A_X \cdot Y) - A_X \cdot A_Y \cdot V$. Thus we obtain

$$S(X, Y) = -\operatorname{div}(A_X \cdot Y) - \operatorname{trace}(A_X A_Y).$$

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[2] After the completion of this paper, there appeared a paper by Robert Hermann, *Totally geodesic orbits of groups of isometries* (Lincoln Laboratory, MIT, June 1960). He makes use of formulas which are essentially the same as ours.

The formula in Lemma 1 follows by taking $Y=X$.

PROOF OF LEMMA 2. We recall that $Z\cdot g(X, Y)=g(\nabla_Z X, Y)+g(X, \nabla_Z Y)$ for arbitrary vector fields X, Y and Z (this is an infinitesimal expression of the fact that the parallel displacement of the Riemannian connection is isometric). Applying this formula, we have

$$V\cdot\phi = (1/2)V\cdot g(X,X) = g(\nabla_V X, X) = g(-A_X\cdot V, X) = g(V, A_X\cdot X),$$

since A_X is skew-symmetric when X is a Killing vector field. We then obtain

$$V^2\phi = V\cdot g(V, A_X X) = g(\nabla_V V, A_X\cdot X) + g(V, \nabla_V(A_X)) = g(V, \nabla_V(A_X)),$$

since $\nabla_V V=0$ by assumption.

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