

A REMARK ON ONE-SIDED APPROXIMATION¹

L. C. EGGAN AND IVAN NIVEN

It is well-known that given any irrational ξ there exist infinitely many rational numbers a/b such that

$$(1) \quad 0 < \frac{a}{b} - \xi < \frac{1}{cb^2}$$

with $c = 1$. All the odd convergents to the continued fraction expansion of ξ , for example, are solutions of (1) with $c = 1$. We examine here the following questions: what is the best possible value of c such that for every irrational ξ the inequalities (1) have (i) infinitely many rational solutions a/b , and (ii) at least one rational solution a/b ? The answer is $c = 1$ in both cases. The answer to question (i) is implied by a result of B. Segre [1, p. 361, Theorem 14], which is stated below as Theorem 2. The answer to question (ii) is given in Theorem 1 below. Theorems 1 and 2 are related but neither implies the other. In proving Theorem 1 we use some results of R. M. Robinson [2; 3], which we also apply to give a short proof of Segre's theorem.

THEOREM 1. *For any $c > 1$ there are uncountably many irrationals ξ for which (1) has no rational solutions a/b . Analogously, for any $c > 1$ there are uncountably many irrationals ξ for which $0 < \xi - a/b < 1/(cb^2)$ has no rational solutions.*

To prove the theorem, we consider the convergents $a_n/b_n = [q_0, q_1, \dots, q_n]$ to the continued fraction expansion $[q_0, q_1, q_2, \dots]$ of an irrational ξ . Let ξ be any irrational such that $q_n = 1$ for n even, $q_n > 2(c-1)^{-1}$ for n odd. Thus we have uncountably many ξ , and we prove that for any such ξ the inequality (1) has no solutions. Defining λ_n by the equation

$$\left| \frac{a_n}{b_n} - \xi \right| = \frac{1}{\lambda_n b_n^2}$$

we know from the theory of continued fractions that $\lambda_n = \beta_n + 1/\alpha_n$, where

$$\beta_n = [q_{n+1}, q_{n+2}, q_{n+3}, \dots], \quad \alpha_n = [q_n, q_{n-1}, \dots, q_1].$$

The even convergents, being less than ξ , do not satisfy (1). Any odd convergent, i.e., a_n/b_n with n odd, has a corresponding λ_n satisfying

Received by the editors August 1, 1960.

¹ Work supported in part by the Office of Naval Research.

$$\lambda_n = \beta_n + \frac{1}{\alpha_n} < q_{n+1} + \frac{1}{q_{n+2}} + \frac{1}{q_n} < 1 + \frac{c-1}{2} + \frac{c-1}{2} = c.$$

Hence the odd convergents do not satisfy (1).

Now Robinson [2, p. 354] has proved two lemmas that imply that if a secondary convergent to ξ satisfies (1), then there is a convergent that also satisfies (1). It follows that since no convergent to ξ satisfies (1), no secondary convergents satisfy (1). But Robinson [3, p. 355] has also established that every approximation a/b to an irrational ξ with an error less than $1/b^2$ is either a convergent or a secondary convergent. Hence (1) has no solutions for the particular numbers ξ under discussion.

To prove the other part of Theorem 1, namely for approximations to the left of ξ , we define ξ as any irrational with $q_n=1$ for n odd, $q_n > 2(c-1)^{-1}$ for n even.

Finally we prove Segre's result, which is stated in contrapositive form.

THEOREM 2 (SEGRE). *An irrational $\xi = [q_0, q_1, q_2, \dots]$ has the property that there exist infinitely many rationals a/b satisfying (1) for some $c > 1$ if and only if $q_{2n} \geq 2$ for infinitely many n or q_{2n+1} is bounded for infinitely many n .*

First we suppose that ξ has the property stated for some fixed $c > 1$, so that there are infinitely many approximations a/b satisfying (1). By the previously cited result of Robinson [3, p. 355] every one of these approximations is either a convergent or a secondary convergent. Furthermore the lemmas of Robinson [2, p. 354] imply that there are infinitely many convergents, odd convergents of course, of ξ that satisfy (1). If $q_{2n} = 1$ for n sufficiently large, and if an odd convergent a_{2n-1}/b_{2n-1} is a solution of (1), then we have

$$c < \lambda_{2n-1} < q_{2n} + \frac{1}{q_{2n+1}} + \frac{1}{q_{2n-1}} = 1 + \frac{1}{q_{2n+1}} + \frac{1}{q_{2n-1}}.$$

Thus one of q_{2n+1} and q_{2n-1} is less than $2(c-1)^{-1}$, and so we have infinitely many q_j with j odd bounded by $2(c-1)^{-1}$.

Conversely, suppose (first hypothesis) that $q_{2n} \geq 2$ for infinitely many $n \geq 1$ or (second hypothesis) that q_{2n+1} is bounded, say by K , for infinitely many $n \geq 1$. Then for any such n , since

$$\lambda_{2n-1} > \beta_{2n-1} > q_{2n} + \frac{1}{q_{2n+1} + 1},$$

we have

$$\lambda_{2n-1} > 2 \text{ (first hypothesis),}$$

or

$$\lambda_{2n-1} > 1 + (K + 1)^{-1} \text{ (second hypothesis).}$$

Hence, we have infinitely many solutions of (1) with $c=2$ or $c=1 + (K+1)^{-1}$.

REFERENCES

1. B. Segre, *Lattice points in infinite domains, and asymmetric Diophantine approximations*, Duke Math. J. vol. 12 (1945) pp. 337-365.
2. R. M. Robinson, *Unsymmetrical approximation of irrational numbers*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 351-361.
3. ———, *The approximation of irrational numbers by fractions with odd or even terms*, Duke Math. J. vol. 7 (1940) pp. 354-359.

UNIVERSITY OF OREGON

A NOTE ON SUBSERIES CONVERGENCE

CHARLES W. McARTHUR¹

1. **Introduction.** Throughout this paper X will denote a Banach space (B -space), X^* its normed conjugate space, and $\sum_{i=1}^{\infty} x_i$ a series in X . A series $\sum_{i=1}^{\infty} x_i$ is *subseries convergent* if and only if

(A) Each subseries of $\sum_{i=1}^{\infty} x_i$ converges. Additional conditions, (B) through (H), are introduced below. The main purpose of this note is to give new and simple proofs of the equivalence of each of (B), (G), and (H) to (A).

The equivalence of (A) and (B), the Orlicz-Pettis theorem, was first stated and proved by Orlicz [11, Satz 2] for weakly complete X . It was stated by Banach [2, p. 240] for general X . The first proof of the equivalence of (A) and (B) for X a B -space was given by Pettis [12, Theorem 2.32]. Conditions (G) and (H) were suggested by Gelfand [8] as definitions of "strong unconditional convergence" of $\sum_{i=1}^{\infty} x_i$ and he [8, Satz 5a, p. 245] stated their equivalence. The equivalence of (H) and (A) is shown in [9]. Thus, the equivalence of (A), (B), (G) and (H) is well known.

Presented to the Society, November 18, 1960; received by the editors September 13, 1960.

¹ This work was supported by a Summer Research Grant from the Research Council of the Florida State University.