ON COMPONENTS IN SOME FAMILIES OF SETS[1]

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1. **Introduction.** Helly's well-known theorem [3] states that all the members of a family $\mathcal{C}$ of compact convex subsets of the Euclidean n-space E^n have a point in common provided every $n+1$ members of $\mathcal{C}$ have a common point. On the other hand (Motzkin, cf. Hadwiger-Debrunner [2] for further reference), there exists no (finite) number h with the following property: If $\mathcal{K}$ is a family of subsets of E^n (even of E^1) such that each member of $\mathcal{K}$ is the union of at most two disjoint, compact, convex sets, and such that every h members of $\mathcal{K}$ have a common point, then all the members of $\mathcal{K}$ have a common point.

A consideration of the examples which establish the nonexistence of h led to the idea that there might exist theorems of Helly's type for such families $\mathcal{K}$ if an additional condition is imposed on $\mathcal{K}$: the intersection of any two members of $\mathcal{K}$ should also be representable as the union of at most two disjoint, compact, convex sets. The present paper contains a theorem in this direction together with related results on families $\mathcal{K}$ whose elements are disjoint unions of members of another family $\mathcal{C}$.

In §2 we give the definitions of the properties we consider, and the statements of our main results. The proofs follow in §3. Remarks, examples, and counter-examples are given in §4.

2. **Definitions and results.** We shall deal mainly with families of subsets of some set, on whose nature nothing is assumed.

For a set A or an ordinal μ we denote by card A resp. card μ the corresponding cardinal. Thus, for a family of sets $\mathcal{C}=\{C_\alpha:\alpha\in A\}$ we have card $\mathcal{C}=$card A. The letter ω is used only for initial ordinals.

For a family of sets $\mathcal{C}=\{C_\alpha:\alpha\in A\}$ we put $\pi\mathcal{C}=\bigcap_{\alpha\in A} C_\alpha$ and $\sigma\mathcal{C}=\bigcup_{\alpha\in A} C_\alpha$.

We define $K=C_1+C_2$ to be an abbreviation for the statement "$K=C_1\cup C_2$ and $C_1\cap C_2=\emptyset$." Similarly, for $\mathcal{C}=\{C_\alpha:\alpha\in A\}$, we write $K=\sum_{\alpha\in A} C_\alpha=\Sigma\mathcal{C}$ for "$K=\sigma\mathcal{C}$ and $C_\alpha\cap C_\beta=\emptyset$ for all α, $\beta\in A$ with $\alpha\neq\beta$."

If $K=\Sigma\mathcal{C}$, each member of $\mathcal{C}$ is a *component* of K and $\Sigma\mathcal{C}$ is a *decomposition* of K.

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For any family $\mathcal{C}$ and any cardinal γ let $[\mathcal{C}]_\gamma = \{\Sigma\mathcal{C}' : \mathcal{C}' \subset \mathcal{C}, \text{card } \mathcal{C}' < \gamma + 1\}$ and $[\mathcal{C}] = \{\Sigma\mathcal{C}' : \mathcal{C}' \subset \mathcal{C}\}$. For $K \in [\mathcal{C}]$ let $c(K) = \min\{\text{card } \mathcal{C}' : K = \Sigma\mathcal{C}', \mathcal{C}' \subset \mathcal{C}\}$.

This paper deals with some properties of families of sets which we proceed to define.

Definition 1. A family $\mathcal{C}$ is *γ-intersectional* (for a finite or infinite cardinal $\gamma \geqq 1$) if for every subfamily $\mathcal{C}' \subset \mathcal{C}$ with card $\mathcal{C}' < \gamma + 1$ we have $\pi\mathcal{C}' \in \mathcal{C}$. The family $\mathcal{C}$ is *intersectional* if it is γ-intersectional for every $\gamma \geqq 1$.

Obviously, if $\gamma^* \leqq \gamma$ and $\mathcal{C}$ is γ-intersectional, it is γ^*-intersectional as well. Every family is 1-intersectional; every 2-intersectional family is $\aleph_0$-intersectional.

Definition 2. A family $\mathcal{C}$ is *γ-nonadditive* (for a finite or infinite cardinal $\gamma \geqq 2$) if for every subfamily $\mathcal{C}' \subset \mathcal{C}$, with $\emptyset \notin \mathcal{C}'$ and $1 < \text{card } \mathcal{C}' < \gamma + 1$, such that $\Sigma\mathcal{C}'$ is defined, we have $\Sigma\mathcal{C}' \notin \mathcal{C}$. The family $\mathcal{C}$ is *nonadditive* if it is γ-nonadditive for every $\gamma \geqq 2$.

Examples. The family of all closed [open] subsets of E^n is intersectional [$\aleph_0$-intersectional]. The family of all connected and open [compact] subsets of E^n is nonadditive [$\aleph_1$-nonadditive; see [4]]. In the set of ordinals $\{\alpha : \alpha < \omega, \text{card } \omega = k\}$, for any $k > \aleph_0$, all segments of the form $[\alpha, \beta]$ or $[\beta, \omega)$, where α, β are limit-ordinals, form a family $\mathcal{S}$ which is intersectional and nonadditive. For any set S with card $S = k \geqq \aleph_0$ the family of all subsets of S with complements of cardinal less than k is $\aleph_0$-intersectional and nonadditive.

Definition 3. A family $\mathcal{C}$ has the *Helly property* of *order h* with *limit γ* (h, γ cardinals with $2 \leqq h < \gamma$) if for each subfamily $\mathcal{C}' \subset \mathcal{C}$, with card $\mathcal{C}' < \gamma + 1$, the condition "$\pi\mathcal{C}^* \neq \emptyset$ for all $\mathcal{C}^* \subset \mathcal{C}'$, with card $\mathcal{C}^* < h + 1$" implies $\pi\mathcal{C}' \neq \emptyset$. The family $\mathcal{C}$ has the *unlimited Helly property* of order h if it has the Helly property of order h with limit γ for every $\gamma > h$.

Examples. The family of all compact subsets of any topological space has the unlimited Helly property of order $\aleph_0$. The family of convex subsets of E^n has the Helly property of order $n+1$ with limit $\aleph_0$; that of compact convex subsets has the unlimited Helly property of order $n+1$ (Helly's theorem). The family of all closed segments $[\alpha, \beta]$ of a well-ordered set has the unlimited Helly property of order 2; if segments $[\alpha, \mu)$, for a limit ordinal μ, are included, the family has the Helly property of order 2 with limit card μ.

The first theorem gives a criterion for the uniqueness of the decomposition of K.

Theorem 1. *Let $\mathcal{C} = \{C_\alpha : \alpha \in A\}$ be 2-intersectional and γ-nonadditive, and $K \in [\mathcal{C}]_\gamma$. If $K = \sum_{\alpha' \in A'} C_{\alpha'}$ with $A' \subset A$*, card $A' < \gamma + 1$, *and* $C_{\alpha'} \neq \emptyset$ *for all* $\alpha' \in A'$, *and if* $K = \sum_{\alpha'' \in A''} C_{\alpha''}$ *with* $A'' \subset A$, card $A'' < \gamma + 1$, *and* $C_{\alpha''} \neq \emptyset$ *for all* $\alpha'' \in A''$, *then there exists a one-to-one map* ϕ *from* A' *onto* A'' *such that* $C_{\alpha'} = C_{\phi(\alpha')}$ *for all* $\alpha' \in A'$. *In other words, the components of K are uniquely determined.*

As an immediate corollary we have:

COROLLARY. *Let* $\mathcal{C}$ *be 2-intersectional and* γ*-nonadditive, and let* $K \in [\mathcal{C}]_n$ (*i.e.*, $c(K) \leqq n$), *where n is a finite cardinal and* $\gamma \geqq n$. *Let* $K^* \in [\mathcal{C}]_\gamma$, $K \subset K^*$, *and let some n different components of* K^* *each have a nonempty intersection with K. Then different components of K are contained in different components of* K^*, *and, in particular*, $c(K) = n$.

Obvious examples show that the corollary may fail for infinite n.

The next theorem shows that $[\mathcal{C}]_\gamma$ is, in a sense, weakly intersectional: if the intersections of all members of certain subfamilies of $\mathcal{K} \subset [\mathcal{C}]_\gamma$ belong to $[\mathcal{C}]_\gamma$, then for each subfamily of $\mathcal{K}$ the intersection of its members belongs to $[\mathcal{C}]_\gamma$.

THEOREM 2. *Let* $\mathcal{C}$ *be* γ*-intersectional and* γ'*-nonadditive*, $\mathcal{K} \subset [\mathcal{C}]_\gamma$ *and* $\pi\mathcal{K} \in [\mathcal{C}]_{\gamma'}$. *Then there exists a subfamily* $\mathcal{K}' \subset \mathcal{K}$, *with* $1 +$card $\mathcal{K}' \leqq c(\pi\mathcal{K})$, *such that different components of* $\pi\mathcal{K}$ *are contained in different components of* $\pi\mathcal{K}'$; *in particular*, $c(\pi\mathcal{K}') \geqq c(\pi\mathcal{K})$.

A result of Helly's type for members of $[\mathcal{C}]_2$ is given by

THEOREM 3. *Let* $\mathcal{C}$ *be* γ*-intersectional and* $\aleph_0$*-nonadditive, with the Helly property of order h and limit* γ^*, $\gamma^* \geqq \aleph_0 > h$. *Let* $\mathcal{K} \subset [\mathcal{C}]_2$ *be such that* card $\mathcal{K} < \gamma + 1$ *and* $K' \cap K'' \in [\mathcal{C}]_2$ *for all* K', $K'' \in \mathcal{K}$. *Then* $\mathcal{K}$ *has the Helly property of order* $2h$ *with limit* γ^*.

3. **Proofs.**

PROOF OF THEOREM 1. Obviously

$$K = \sum_{\alpha' \in A';\, \alpha'' \in A''} (C_{\alpha'} \cap C_{\alpha''})$$

is a decomposition of K. If for each $\alpha' \in A'$ and each $\alpha'' \in A''$ either $C_{\alpha'} \cap C_{\alpha''} = \emptyset$ or $C_{\alpha'} \cap C_{\alpha''} = C_{\alpha'}$, the theorem is proved. Assume on the contrary that there exists an $\alpha_0' \in A'$ and an $\alpha_0'' \in A''$ such that $C_{\alpha_0'} \cap C_{\alpha_0''}$ is neither $\emptyset$ nor $C_{\alpha_0'}$. Let $A_0'' = \{\alpha'' \in A'' : C_{\alpha_0'} \cap C_{\alpha''} \neq \emptyset\}$. Then $2 \leqq$card $A_0{}' < \gamma + 1$ and $C_{\alpha_0'} = C_{\alpha_0'} \cap K = C_{\alpha_0'} \cap \sum_{\alpha'' \in A''} C_{\alpha''} = \sum_{\alpha'' \in A_0'{}'} (C_{\alpha''} \cap C_{\alpha_0'})$, in contradiction to the γ-nonadditivity of $\mathcal{C}$.

PROOF OF THEOREM 2. (i) Let $c(\pi\mathcal{K}) \geqq 2$. Then there exist points x_1 and x_2 contained in different components C_1^*, C_2^* of $K^* = \pi\mathcal{K}$. For some $K_0 \in \mathcal{K}$ the points x_1 and x_2 are contained in different components of K_0; indeed, otherwise there would for each $K \in \mathcal{K}$ exist a component C' of K with $x_1, x_2 \in C'$. Now $C = \pi\{C' : K \in \mathcal{K}\} \in \mathcal{C}$ but, on the other hand, $C = C \cap K^* \supset (C \cap C_1^*) + (C \cap C_2^*)$, and none of the components is empty (since $x_i \in C \cap C_i^*$), contradicting the γ'-non-additivity of $\mathcal{C}$. If $c(K^*) = 2$, it follows at once from the corollary to Theorem 1 that different components of K^* are contained in different components of K_0.

(ii) We now assume that $c(K^*) = n$ is finite, $n > 2$, and that the theorem is proved for all n' with $n' < n$. We start as in (i) with a set $K_0 = \sum_{\nu \in N} C_\nu \in \mathcal{K}$, where card $N = c(K_0) \geqq 2$, such that $C_1 \cap K^* \neq \emptyset$ and $C_2 \cap K^* \neq \emptyset$. Let $q_\nu = c(K^* \cap C_\nu) \geqq 0$ for $\nu \in N$. By Theorem 1 we have

$$(*) \qquad \sum_{\nu \in N} q_\nu = c(K^*) = n.$$

This implies that $N_0 = \{\nu \in N : q_\nu > 0\}$ is finite and contains at most n elements. Let us assume that $N_0 = \{1, 2, \cdots, t\}$ and that the components of K_0 are labeled in such a way that $q_\nu \geqq 2$ for $1 \leqq \nu \leqq s$, and $q_\nu = 1$ for $s < \nu \leqq t$. If $s = 0$, then (*) implies $t = n$, and by the corollary to Theorem 1 the n components of K_0 contain the n components of K^*, as claimed. Thus we are left with the case $s \geqq 1$; then $2 \leqq t < n$, $q_1 \geqq 2$ and, by the choice of K_0, $q_2 \geqq 1$; therefore, by (*), $q_\nu < n$ for all $\nu \in N_0$. This allows us to apply the inductive assumption to each of the s families $\mathcal{K}_\nu = \{C_\nu \cap K : K \in \mathcal{K}\}$, $1 \leqq \nu \leqq s$. It follows that for each ν, with $1 \leqq \nu \leqq s$, there exists a subfamily $\mathcal{K}_\nu' \subset \mathcal{K}_\nu$, containing $p_\nu \leqq q_\nu - 1$ members, such that the different components of $C_\nu \cap K^*$ are contained in different components of $\pi \mathcal{K}_\nu'$. The family $\mathcal{K}' = \{K_0\} \cup (\bigcup_{\nu=1}^{s} \mathcal{K}_\nu')$ satisfies all the conditions of the theorem. Indeed, different components of K^* are, by the corollary to Theorem 1, contained in different components of $\pi \mathcal{K}'$; but on the other hand, $\mathcal{K}'$ contains only $1 + \sum_{\nu=1}^{s} p_\nu \leqq 1 - s + \sum_{\nu=1}^{s} q_\nu = 1 - s + n - (t - s) = n + 1 - t \leqq n - 1 < c(K^*)$ members.

(iii) There remains the case in which $k = c(K^*)$ is infinite. Let ω be the initial ordinal of k and let $K^* = \pi \mathcal{K} = \sum_{\nu < \omega} C_\nu^*$. For each $\nu < \omega$ let $x_\nu \in C_\nu^*$. As in (i), for each pair $\nu, \mu < \omega$ with $\nu \neq \mu$ there exists some $K_{\nu,\mu} \in \mathcal{K}$ such that x_ν and x_μ are contained in different components of $K_{\nu,\mu}$. Let $\mathcal{K}' = \{K_{\nu,\mu} : \nu, \mu < \omega\}$. Then card $\mathcal{K}' \leqq (\text{card } \omega)^2 = k$. For the family $\mathcal{K}'$ we have $c(\pi \mathcal{K}') \geqq k$ since x_ν and x_μ belong to different components of $\pi \mathcal{K}'$. By an argument similar to that used in the proof of Theorem 1 it follows that different components of K^* are contained in different components of $\pi \mathcal{K}'$. This ends the proof of Theorem 2.

PROOF OF THEOREM 3. For some fixed h assume the theorem false; let k be the minimal cardinal for which there exists a family with card $\mathcal{K}=k$ contradicting the theorem.

(i) Assume k finite. Then for each $K^* \in \mathcal{K}$ we have $\pi\{K\in\mathcal{K}: K\neq K^*\}\neq\emptyset$. Let $\mathcal{K}_i=\{K\in\mathcal{K}: c(K)=i\}$ for $i=1$, 2, and let $K=C_1+C_2$ for all $K\in\mathcal{K}_2$. We assume that $\mathcal{K}$ is chosen in such a way that card $\mathcal{K}_1+2$ card $\mathcal{K}_2$ (the total number of components of members of $\mathcal{K}$) is minimal. This implies that for each $K'\in\mathcal{K}_2$ and $i=1$, 2, there exists a $K^0=K^0(C_i')=K^0(K', i)\in\mathcal{K}$ such that $\pi\{K\in\mathcal{K}: K\neq K^0\}\subset C_i'$.

We shall show that $C_i'\cap K\neq\emptyset$ for all $K'\in\mathcal{K}_2$, $K\in\mathcal{K}$, and $i=1, 2$. Let us assume, to the contrary, that there exists $K'\in\mathcal{K}_2$, $K_0\in\mathcal{K}$ and $i=1$ or 2 such that $C_i'\cap K_0=\emptyset$. (Without loss of generality we shall assume $i=1$.) Since $\emptyset\neq\pi\{K\in\mathcal{K}: K\neq K^0(K', 1)\}\subset C_i'$, it follows that $K_0=K^0=K^0(K', 1)$. Then $C_1'\cap K\neq\emptyset$ for all $K\neq K^0$; also $C_2'\cap K\neq\emptyset$ for all $K\in\mathcal{K}$, since otherwise $K'\cap K\cap K^0\subset(C_1'\cap K^0)\cup(C_2'\cap K)=\emptyset$ would contradict the assumption that any $3<4\leqq 2h$ members of $\mathcal{K}$ have a nonempty intersection. Therefore, for each $K\neq K^0$, $c(K'\cap K)=2$; hence, for some component C_j of K we have $K\cap C_2'=C_j\cap C_2'$. Now

$$\begin{aligned}\pi\{C_j: K\in\mathcal{K}, K\neq K^0\} &= C_2'\cap\pi\{C_j: K\in\mathcal{K}, K\neq K^0\}\\ &= C_2'\cap\pi\{K\in\mathcal{K}: K\neq K^0\}\subset C_1'\cap C_2'=\emptyset.\end{aligned}$$

Since $\mathcal{C}$ has the Helly property of order h it follows that for some subset $\mathcal{K}_0$ of $\mathcal{K}$, such that $K^0\notin\mathcal{K}_0$ and with card $\mathcal{K}_0=h_0\leqq h$, we have $\pi\{C_j: K\in\mathcal{K}_0\}=\emptyset$. For the family $\mathcal{K}^*=\{K', K^0\}\cup\mathcal{K}_0$ we have therefore $\pi\mathcal{K}^*\subset(C_1'\cap K^0)\cup(C_2'\cap\pi\mathcal{K}_0)=\emptyset$, although card $\mathcal{K}^*\leqq h_0+2\leqq h+2\leqq 2h$. This contradiction establishes our assertion.

Next, let $K^*\in\mathcal{K}_2$ be chosen arbitrarily. For each $K\in\mathcal{K}_2$ it follows from the above and from $c(K^*\cap K)\leqq 2$ that $c(K^*\cap K)=2$ and that different components of K intersect different components of K^*. Let the components of K be re-labeled, if necessary, in such a way that $C_i^*\cap C_i\neq\emptyset$ for $i=1$, 2. We claim that for all K', $K''\in\mathcal{K}_2$ we have $C_i'\cap C_i''\neq\emptyset$, $i=1, 2$. Indeed, otherwise we would have (since each component of one set intersects every other set), $C_1'\cap C_1''=C_2'\cap C_2''=\emptyset$, and therefore $K^*\cap K'\cap K''=\emptyset$, which is impossible. Thus, for any K', $K''\in\mathcal{K}_2$,

$$C_i'\cap C_j''\begin{cases}=\emptyset & \text{if } i\neq j\\ \neq\emptyset & \text{if } i=j.\end{cases}$$

Now we consider the families $\mathcal{C}_i=\mathcal{K}_1\cup\{C_i: K\in\mathcal{K}_2\}$ for $i=1$, 2. The assumption $\pi\mathcal{K}=\emptyset$ implies that $\pi\mathcal{C}_i=\emptyset$ for $i=1$, 2. Since $\mathcal{C}_i\subset\mathcal{C}$, there exist h or less members of $\mathcal{C}_i$ whose intersection is empty, $i=1, 2$. But then the intersection of the corresponding members of $\mathcal{K}$ is also empty, although it involves at most $2h$ members of $\mathcal{K}$. The contradiction reached proves the theorem for finite k.

(ii) Let k be infinite, $k<\gamma^*$, and the theorem true for all families with less than k members. Let ω be the initial ordinal of k, let A be the set of ordinals $A=\{\alpha:\alpha<\omega\}$, and let $\mathcal{K}=\{K_\alpha:\alpha<\omega\}$. By the inductive assumption we have $\bigcap_{\alpha<\mu} K_\alpha\neq\emptyset$ for each $\mu<\omega$. If for some K_α one of its components does not intersect some K_β, we omit this component and take the other component to be the new K_α. By the inductive assumption, the new K_α satisfy $\bigcap_{\alpha<\mu} K_\alpha\neq\emptyset$ for all $\mu<\omega$. From here on we proceed as in the final part of (i): we re-label (if necessary) the components of some K_α with $c(K_\alpha)=2$, construct the families $\mathcal{C}_i$ and derive a contradiction from the assumption that $\bigcap_{\alpha<\omega} K_\alpha=\emptyset$. This terminates the proof of Theorem 3.

4. **Remarks.** 1. Theorem 2 fails if card $\pi\mathcal{K}$ is infinite and $\mathcal{K}'$ is assumed to satisfy card $\mathcal{K}'<$card $\pi\mathcal{K}$. E.g., starting from the family $\mathcal{S}$ (preceding Definition 3), with card $\omega=k>\aleph_0=$ card ω_0, let $\mathcal{K}=\{[\omega_0, \alpha]\cup[\alpha+\omega_0, \omega): \alpha$ limit ordinal $<\omega\}$. Then $c(\pi\mathcal{K})=k$, but the intersection of any $k'<k$ members of $\mathcal{K}$ has only k' components. Similar examples are easily found for $c(\pi\mathcal{K})=\aleph_0$.

2. Probably the most interesting immediate application of Theorem 3 is to convex sets in E^n. To satisfy the condition of nonadditivity we may consider, e.g., families consisting only of closed (or only of open) convex sets. The following example shows that Theorem 3 does not hold if $\mathcal{C}$ is, e.g., the family of all convex sets in E^2. (Simple examples of a similar nature show the necessity of nonaddivitity assumptions in Theorem 2.) Let D denote a closed disc with center 0. Let K_0 be obtained from D by deleting 0. Let x_i, $i=1, 2, \cdots, 6$, be equidistant points on the boundary of D, ($x_i=x_{i+6}$). For each i, $1\leqq i\leqq 6$, let K_i be obtained from D by deleting the open small arc of Bd D determined by x_{i-1} and x_{i+1}, and the open sector determined by these two points and 0. Then each K_i, $0\leqq i\leqq 6$, as well as the intersection of any two K_i, is the disjoint union of two convex sets, and any six K_i have a nonempty intersection. Nevertheless, $\bigcap_{i=0}^{6} K_i=\emptyset$. As is easily verified, the same reasoning applies to the case where 7 or 8 equidistant points are chosen on Bd D. We conjecture that for the family of all convex sets in E^2 a result analogous to Theorem 3 holds, with 9 instead of $2h$.

3. The following statement (with obvious refinements) is conjectured: If $\mathcal{C}$ is an intersectional and nonadditive family with un-

limited Helly property of order h and if $\mathcal{K}\subset[\mathcal{C}]_n$ is such that the intersection of any 2, 3, $\cdots$, n members of $\mathcal{K}$ also belongs to $[\mathcal{C}]_n$ then $\mathcal{K}$ has the unlimited Helly property of order nh. Simple examples show that $nh-1$ may not be substituted for nh in this conjecture. If $\mathcal{C}$ is the family of segments in E^1, the conjecture is easily provable.

4. Let $\mathcal{C}^{(n)}$ denote the family of all compact, convex subsets of E^n; in [1], a function $\Delta(K)$, with $0\leqq\Delta(K)\leqq+\infty$, was defined for all compact sets $K\subset E^n$ in such a way that $\Delta(K)<\infty$ if and only if $K\in[\mathcal{C}^{(n)}]_{\aleph_0}$. Theorem 2 of [1] may be formulated as follows: For any finite $n\geqq 1$ and real $d<\infty$ there exists a finite $h=h(n, d)$ such that the family $\{K\in[\mathcal{C}^{(n)}]_{\aleph_0}: \Delta(K)\leqq d\}$ has the unlimited Helly property of order h. By applying the methods of [1] it may be shown that for each finite $n\geqq 1$ and $d<\infty$ there exists a finite $k=k(n, d)$ such that $\Delta(K)\leqq d$ implies $K\in[\mathcal{C}^{(n)}]_k$.

REFERENCES

1. B. Grünbaum, *A variant of Helly's theorem*, Proc. Amer. Math. Soc. vol. 11 (1960) pp. 517–522.

2. H. Hadwiger and H. Debrunner, *Ausgewählte Einzelprobleme der kombinatorischen Geometrie in der Ebene*, Enseignement Math. vol. 1 (1955) pp. 56–89.

3. E. Helly, *Ueber Mengen konvexer Körper mit gemeinschaftlichen Punkten*, Jber. Deutsch. Math. Verein. vol. 32 (1923) pp. 175–176.

4. W. Sierpiński, *Un théorème sur les continus*, Tôhoku Math. J. vol. 13 (1918) pp. 300–303.

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