ON THE SPECTRUM OF A CONTRACTION[1]

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1. **Introduction.** In this note we present several results on the spectrum of a contraction. The first is an extension to the approximate point spectrum of a result of Nagy and Foias, on the relation of the point spectrum of a contraction and that of its unitary dilation, which has several corollaries. The second is a simple solution to a problem in spectral mapping raised in [2]. Finally we have a result on the point spectrum of a class of contractions discussed in [3]. For the background on unitary dilations see [4] or [5].

2. **Arbitrary contractions.**[2] In Theorem 1 of [6] it is shown that the set of eigenvalues of modulus 1 of a contraction A coincides with that of its unitary dilation U. Less is true for the approximate point spectrum $\Sigma_{ap}A$. (See [1] for the definition of Σ_{ap}.)

PROPOSITION. *Let A be a contraction on a Hilbert space H and let U be a unitary dilation on a (larger) space K. Then $\mu = e^{ix} \in \Sigma_{ap}A$ if and only if $\mu \in \Sigma_{ap}U$ with approximate eigenvectors in H.*[3]

(Thus, if $\mu \in \Sigma_{ap}U$, $|\mu| = 1$, but the approximate eigenvectors are not in H, then $\mu \notin \Sigma_{ap}A$.)

PROOF. Let P be the projection of K onto H. If there are unit vectors $x_n \in H$ with $\|Ux_n - \mu x_n\| \to 0$ as $n \to \infty$, then $\|Ax_n - \mu x_n\| = \|PUx_n - \mu P x_n\| \leqq \|Ux_n - \mu x_n\| \to 0$ as $n \to \infty$, so that $\mu \in \Sigma_{ap}A$. For the converse, there is clearly no loss of generality in taking $\mu = 1$, and we suppose there are unit vectors $x_n \in H$ such that

$$\|Ax_n - x_n\| \leqq 1/n, \qquad n = 1, 2, \cdots,$$

from which it follows that $\|Ax_n\| \geqq 1 - 1/n$. Again let P be the projection of K onto H, and write $H^\perp$ for the orthogonal complement of H in K. Now $Ux_n = u_n + v_n$, with $u_n \in H$, $v_n \in H^\perp$, and $\|u_n\|^2 + \|v_n\|^2 = \|Ux_n\|^2 = \|x_n\|^2 = 1$. Since $u_n = PUx_n = Ax_n$, we have

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[2] We are grateful to the referee for simplifications of the arguments in this and the following paragraph.

[3] By a unitary dilation of an operator A on H is meant a unitary operator U on a space $K \supset H$ such that $PUx = Ax$ for all $x \in H$, where P is the projection of K onto H. In [4; 5] a unique minimal such dilation is studied, but for present purposes minimality is irrelevant.

$$1 = \|Ax_n\|^2 + \|v_n\|^2 \geqq 1 - \frac{2}{n} + \frac{1}{n^2} + \|v_n\|^2,$$

so that

$$\|v_n\|^2 \leqq \frac{2}{n} - \frac{1}{n^2}.$$

The components in H and $H^\perp$ of $Ux_n - x_n$, the vector whose norm is to be estimated, are $Ax_n - x_n$ and v_n, respectively, as is clear, and so

$$\|Ux_n - x_n\|^2 = \|Ax_n - x_n\|^2 + \|v_n\|^2 \leqq \frac{1}{n^2} + \frac{2}{n} - \frac{1}{n^2} = \frac{2}{n},$$

by the displayed inequalities, whence $\|Ux_n - x_n\| \leqq (2/n)^{1/2}$ and the proof is complete.

[ADDED IN PROOF. Professor Sz.-Nagy has remarked in a private communication that the proposition may be proved very simply as follows. With notation as above, for $x \in H$ we have

$$\begin{aligned}\|Ux - x\|^2 &= \|Ux\|^2 + \|x\|^2 - 2\,\mathrm{Re}(Ux, x)\\ &= 2\|x\|^2 - 2\,\mathrm{Re}(Tx, x)\\ &= 2\,\mathrm{Re}(x, x - Tx) \leqq 2\|x\|\|x - Tx\|,\end{aligned}$$

and the conclusion follows at once.]

COROLLARY 1. *The approximate eigenvalues of modulus* 1 *of* A^* *are the complex conjugates of those of* A.

PROOF. If $\mu = e^{ix} \in \Sigma_{ap}A$ then $\mu \in \Sigma_{ap}U$ with approximate eigenvectors in H, so that given $\epsilon > 0$ there exists a unit vector $x \in H$ with $\|Ux - \mu x\| < \epsilon$. Hence $\|U^*x - \bar{\mu}x\| < \epsilon$, trivially, and by the proposition again it follows that $\|A^*x - \bar{\mu}x\| < \epsilon$, as was to be shown.

The same result for the point spectrum is given in [**5**, p. 88].

COROLLARY 2. *Near a gap in* ΣU *there can be only residual spectrum of* A.

PROOF. By a gap in ΣU is meant an open arc of the unit circle which lies in the complement of ΣU, and the assertion is that every such gap is contained in a planar open set disjoint from $\Sigma_{ap}A$. The proof is based on the closure of $\Sigma_{ap}A$. Suppose this for the moment. Let G be a gap in ΣU and $e^{i\alpha} \in G$. Then there must be an open circle C_α centered at $e^{i\alpha}$ with $C_\alpha \cap \Sigma_{ap}A = \emptyset$, else $e^{i\alpha}$ would be a limit point of $\Sigma_{ap}A$, hence in $\Sigma_{ap}A$, and therefore by the proposition a member of $\Sigma_{ap}U$, contrary to supposition. The open set required by the corollary is then $\bigcup_\alpha C_\alpha$. We complete the proof by showing that $\Sigma_{ap}A$ is closed, for any bounded A. Let $\lambda_n \in \Sigma_{ap}A$, $\lambda_n \to \lambda$. If $\lambda \notin \Sigma_{ap}A$ then there exists $\epsilon > 0$ such that $\|(A-\lambda I)x\| \geqq \epsilon$ for all unit vectors x. Then $|\lambda - \lambda_n| \doteq \|(A-\lambda I)x - (A-\lambda_n I)x\| \geqq \epsilon - \|(A-\lambda_n I)x\|$, so $\|(A-\lambda_n I)x\| \geqq \epsilon - |\lambda - \lambda_n|$ for all unit vectors x. In particular, if $|\lambda_{n_0} - \lambda| \leqq \epsilon/2$ then $\|(A-\lambda_{n_0} I)x\| \geqq \epsilon/2$ for all unit vectors, so that $\lambda_{n_0} \notin \Sigma_{ap}A$, contrary to supposition.

A side condition such as the one employed in the proposition (that the approximate eigenvectors for U be in H) is seen to be necessary by taking for A any contraction with no spectrum on $\{|z|=1\}$, whereas $\Sigma U \subset \{|z|=1\}$ and $\Sigma U = \Sigma_{ap} U$ (see [1, p. 51]). In particular we know that for such A the approximate eigenvectors of its dilation U cannot be in H.

3. **A spectral mapping problem.** In [2] we studied the preservation of $\Sigma_p A$ under general mappings and noted that in general it is not preserved in the reverse direction (that is, $\alpha \in f^{-1}(\beta)$ need not be an eigenvalue of A when $f(\beta)$ is an eigenvalue of $f(A)$). On the other hand it is trivially clear that if for all function f the number $f(\mu)$ is an eigenvalue of $f(A)$ then μ is an eigenvalue of A. The problem is to find a nonvacuous condition sufficient for preservation of $\Sigma_p A$ in the reverse direction.

Let $C_n(f)$ be the nth Taylor coefficient of f, and write f_t for the function $f_t(s) = f(ts)$.

PROPOSITION. *Let A be a contraction, and f a fixed function analytic for $|z| < 1$. If $f_t(A)x = f_t(\mu)x$ for infinitely many (complex) t converging inside the unit circle, then $A^m x = \mu^m x$, where m is the least $n > 0$ such that $C_n(f) \neq 0$. Conversely, if $A^m x = \mu^m x$ and $C_k(f) = 0$ for $0 \leqq k < m$ then $f_t(A)x = f_t(\mu)x$ for all $|t| < 1$.*

PROOF. By hypothesis $f(z) = \sum_0^\infty C_n(f) z^n$ converges for $|z| < 1$, so $f_t(z) = \sum_0^\infty C_n(f) t^n z^n$ has radius of convergence $r(t) > 1$ for $|t| < 1$. Since $\|A\| \leqq 1$ the operator series $\sum_0^\infty C_n(f) A^n t^n$ converges in norm, for $|t| < 1$, to an operator which we define as $f_t(A)$, so that

$$(f_t(A)x, y) = \sum_0^\infty C_n(f)(A^n x, y) t^n \equiv F(t), \qquad |t| < 1,$$

is an analytic function of t, for each pair x, y of vectors. (This definition of $f_t(A)$ agrees with that of [4], $(f_t(A)x, y) = \int f_t(e^{is}) dF(s)x, y)$, for by uniform convergence the integral is equal to

$$\sum_0^\infty C_n(f) t^n \int e^{ins} d(F(s)x, y) = \sum C_n(f) t^n (A^n x, y),$$

and it is easy to see also that it agrees with the classical definition by the Canchy integral formula.) Similarly $f_t(\mu)(x, y) = (f_t(\mu)x, y)$ may be expanded in the series

$$(f_t(\mu)x, y) = \sum_0^\infty C_n(f)(\mu^n x, y)t^n \equiv G(t), \qquad |t| < 1.$$

Now for the first assertion of the proposition we have by hypothesis that $F = G$ on an infinite set with limit point inside the circle. Since F and G are clearly analytic for $|t| < 1$ we conclude that $F(t) = G(t)$, $|t| < 1$. This means that, for all $n \geqq 0$ and all $y \in H$,

$$c_n(f)(A^n x, y) = c_n(f)(\mu^n x, y),$$

and the assertion now follows by cancellation of $c_n(f)$. The second assertion goes in the same spirit. The hypotheses involve $F(t) = G(t)$ for $|t| < 1$ and therefore $(f_t(A)x, y) = (f_t(\mu)x, y)$ for all y and $|t| < 1$, which yields the conclusion.

4. A contraction A is absolutely continuous if there exists a function $K(t, x, y) \in L_1(0, 2\pi)$ for every pair of vectors x, y, such that

$$(A^{(n)}x, y) = \frac{1}{2\pi}\int_0^{2\pi} e^{int}K(t, x, y)dt$$

for all $n = 0, \pm 1, \cdots$ (here $A^{(-n)} = A^{*n}$) (see [3]). This is a smoothness condition which reflects itself in the spectrum of A as follows:

PROPOSITION. *An absolutely continuous contraction has no eigenvalues of modulus* 1.

PROOF. Let A be absolutely continuous. The representation above for A in terms of K amounts to the assertion that K has the Fourier expansion

$$K(t, x, y) \sim \sum_{-\infty}^{\infty} e^{-int}(A^{(n)}x, y).$$

Now suppose that $Ax = e^{i\beta}x$ for some unit vector x and $0 \leqq \beta \leqq 2\pi$. It then follows from [5, p. 88] that $A^{(n)}x = e^{in\beta}x$ for $n = 0, \pm 1, \pm 2, \cdots$. Hence the Fourier expansion for $K(\cdot, x, x)$ reduces to

$$K(t, x, x) \sim \sum_{-\infty}^{\infty} e^{in\beta}e^{-int}.$$

But $K(\cdot, x, x) \in L_1$, so that its Fourier coefficients $e^{in\beta}$ must tend to 0. This contradiction completes the proof.

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A SUBSTITUTE FOR LEBESGUE'S BOUNDED CONVERGENCE THEOREM

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1. Lebesgue's bounded convergence theorem has become a powerful tool in the theory of linear topological spaces, and recently, for a treatment of weak convergence of sequences or for a proof of Krein's theorem, the tendency is to use it in an essential way.[1] The following is a useful substitute for the bounded convergence theorem stated in the language of linear space theory.

THEOREM 1. *Let C be a compact (or countably compact)*[2] *subset of a (real or complex) linear topological space E, and let $\{f_n\}$ be a sequence of continuous linear functionals on E which is uniformly bounded on C. If, for each x in C, $\lim_n f_n(x) = 0$, then the same equality holds for every x in the closed convex extension of C.*

In case C is compact and Hausdorff, the proof of Theorem 1 may run as follows: Let F be the Banach space of all scalar-valued continuous functions on C with the supremum norm; then there is a linear transformation T on the dual E^* of E into F defined by the equation $T(f) = f | C$. Let x_0 be a point in the closed convex extension of C. Then one can define a bounded functional ϕ on the range of T

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[1] I am indebted to the referee for the remark that, in Dunford and Schwartz [2], Krein's theorem is proved using Riesz-Markoff-Kakutani's theorem but not Lebesgue's bounded convergence theorem. Their proof relies on the theory of integration of vector-valued functions.

[2] A space X is countably compact if each sequence in X has a cluster point.