SURFACE AREA OF A CONVEX BODY UNDER AFFINE TRANSFORMATIONS

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1. **Introduction.** Let K be a nondegenerate convex body in E^n. We will seek requirements on K such that its surface area is minimal compared to that of its affine transforms of the same volume. This problem has been studied for $n=2$ by Behrend [1], Green [2], and Gustin [3].

2. **Necessary and sufficient conditions.** Let S denote the surface of K, $V(K)$ its volume and $S(K)$ its surface area.

THEOREM 1. *Necessary and sufficient conditions that a nondegenerate convex body K in E^n have minimal surface area among its affine transforms of volume $V(K)$ are that*

$$\int_S u_i u_j dS_u = 0 \qquad \text{for } i \neq j$$

and

$$\int_S u_i^2 dS_u = \frac{1}{n} S(K)$$

for $i, j=1, \cdots, n$ where $(u_1, \cdots, u_n)$ is the outer unit vector normal to S at the point with surface area element dS_u.

PROOF. We need only consider central affine transformations of the type (x, y are column vectors of length n)

$$y = Bx, \qquad \det(B) = 1. \tag{1}$$

Suppose that the convex body K_i $(i=1, \cdots, n)$ is transformed into K_i^* by (1), then the linear combination $\lambda_1 K_1 + \cdots + \lambda_n K_n$ is transformed into $\lambda_1 K_1^* + \cdots + \lambda_n K_n^*$. Since volume is preserved, it follows from the definition of mixed volumes [4, p. 38] that

$$V(K_{p_1}, \cdots, K_{p_n}) = V(K_{p_1}^*, \cdots, K_{p_n}^*). \tag{2}$$

Let E be an ellipsoid with supporting function $E(u)$, then

$$nV(K, \cdots, K, E) = \int_S E(u) dS_u, \qquad |u| = 1. \tag{3}$$

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It is clear, then, that our problem is equivalent to keeping K fixed and minimizing $V(K, \cdots, K, E)$ over all ellipsoids with center at the origin and volume

$$\kappa_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}. \tag{4}$$

To show the existence of a minimizing ellipsoid, let U be a solid sphere, of radius $r>0$, contained in K and let $\bar{a}$, of length a, be a segment of maximum length contained in E. By the inclusion property of mixed volumes [4, p. 41, (5)] and the representation [4, p. 45] for the area of the projection of U, we have $V(K, \cdots, K, E) \geqq V(U, \cdots, U, \bar{a}) = ar^{n-1}n^{-1}\kappa_{n-1}$. Consequently, a sequence of ellipsoids E_i of the above type, for which $V(K, \cdots, K, E_i)$ tends to the g.l.b. of (3), is uniformly bounded and by Blaschke's selection theorem [4, p. 34] there exists a subsequence of the E_i which converges to a convex body E^*. The body E^* is necessarily an ellipsoid and, by the continuity of mixed volumes, E^* is a minimizing ellipsoid.

Now, let A^{-1} be a positive definite symmetric matrix with $\det(A^{-1}) = 1$. If E is the ellipsoid with surface

$$x^T A^{-1} x = 1 \tag{5}$$

(the superscript T meaning transpose) then the polar reciprocal of E with respect to the origin has distance function

$$E(u) = \{u^T A u\}^{1/2}. \tag{6}$$

The above is clear if A^{-1} is a diagonal matrix and the general case follows by considering an orthogonal transformation. Consequently, (6) is the supporting function of E, see [4, p. 28], and if E is transformed into E^* by (1) then $E^*(u)$ is found by replacing A in (6) by BAB^T.

Let $F(B)$ be the function of n^2 variables b_{ij} given by

$$F(B) = \int_S E_A(u)dS_u, \qquad |u| = 1, \tag{7}$$

where $E_A(u)$ is given by (6) and $A = BB^T$. Since BB^T is positive semidefinite, (7) is defined for all B. We wish to minimize (7), with the constraint

$$\det(B) = 1. \tag{8}$$

If we equate to zero the partial derivatives of $F(B) + \lambda(1 - \det(B))$ and use (8) we have

$$\left[\frac{\partial F}{\partial b_{ij}}\right] = \lambda(B^T)^{-1}. \tag{9}$$

Also

$$\left[\frac{\partial F}{\partial b_{ij}}\right] = \left[\int_S u_i u_j E_A^{-1}(u)\,dS_u\right]B. \tag{10}$$

Consequently,

$$A^{-1} = \frac{1}{\lambda}\left[\int_S u_i u_j E_A^{-1}(u)\,dS_u\right], \qquad \lambda = \frac{1}{n}\int_S E_A(u)\,dS_u. \tag{11}$$

If A is any positive definite symmetric matrix with det $(A)=1$ which satisfies (11) then the ellipsoid with supporting function (6) will be called a critical ellipsoid of K. It is clear that the unit matrix satisfies (11) if and only if the conditions in Theorem 1 are satisfied. Consequently, a proof of the following result completes the proof of Theorem 1.

THEOREM 2. *If K is a nondegenerate convex body with surface S, then there exists one and only one positive definite symmetric matrix A with* det $(A)=1$ *such that*

$$A^{-1} = \frac{n}{\int_S E_A(u)\,dS_u}\left[\int_S u_i u_j E_A^{-1}(u)\,dS_u\right]$$

where $E_A(u)$ is given by (6).

PROOF. Suppose C is the matrix of an affine transformation of type (1) which carries K into K^*. If $F^*(D)$ is the corresponding function (7) for K^* then

$$F(B) = F^*(CB) \tag{12}$$

for all B. By (2) and (3), this is the case for B with det $(B)=1$ and by the special form of (7) this may be extended to all B. We have then

$$\left[\frac{\partial F}{\partial b_{ij}}\right] = C^T\left[\frac{\partial F^*}{\partial d_{ij}}\right], \qquad D = CB. \tag{13}$$

Consequently, by (11), (13), (10) and (12) we conclude that a critical ellipsoid of K is carried by C into a critical ellipsoid of K^*. We may, therefore, assume that the solid unit sphere is a critical ellipsoid of K and consequently that the conditions in Theorem 1 hold. Let

$$E(a, u) = \left[\sum_1^n a_i^2 u_i^2 \right]^{1/2} \tag{14}$$

and

$$F(a) = \frac{1}{S(K)} \int_S E(a, u) dS_u. \tag{15}$$

The function $F(a)$ is defined for all points a in E^n and has the following properties:

(1) $F(a) > 0$ for $a \neq 0$ and $F(0) = 0$,

(2) $F(\lambda a) = \lambda F(a)$, $\lambda > 0$,

(3) $F(a+b) \leqq F(a) + F(b)$.

To prove property (3), we have by Cauchy's inequality

$$\sum a_i b_i u_i^2 \leqq [(\sum a_i^2 u_i^2)(\sum b_i^2 u_i^2)]^{1/2}$$

and consequently

$$E(a + b, u) \leqq E(a, u) + E(b, u).$$

By [4, pp. 21–22], $F(a)$ is the distance function of a convex body and, by the conditions in Theorem 1, this convex body lies entirely in the closed half-space containing the origin and bounded by the tangent plane $\sum_1^n x_i/n^{1/2} = n^{1/2}$. Now let M be the set of all points a such that $a_i > 0$ and $a_1 \cdots a_n = 1$. By the inequality between the geometric and arithmetic means we have $\sum_1^n a_i/n^{1/2} \geqq n^{1/2}$ for all points in M with equality only if $a_i = 1$ $(i = 1, \cdots, n)$. Consequently, $F(a)$ evaluated over all points in M has a unique absolute minimum for $a = (1, \cdots, 1)$. This completes the proof of Theorem 2 since the conditions in Theorem 1 are invariant under an orthogonal transformation.

3. **The projection body.** Green [2] has shown that for $n = 2$ the conditions in Theorem 1 may be replaced by the requirement that the second Fourier coefficients of the supporting function $H(\cos\theta, \sin\theta) = h(\theta)$ of K vanish. We may generalize this to n-dimensions if in place of the supporting function of K we use the supporting function $\sigma(u)$ of the projection body of K, see [4, p. 45]. This may be viewed as a generalization since for $n = 2$, $\sigma(\cos\theta, \sin\theta) = h(\theta + \pi/2) + h(\theta - \pi/2)$.

THEOREM 3. *If K is a nondegenerate convex body whose projection body has supporting function $\sigma(u)$, then necessary and sufficient conditions that the surface area of K be minimal among its affine transforms of volume $V(K)$ are that*

$$\int_\Omega u_i u_j \sigma(u) d\omega_u = 0 \qquad \textit{for } i \neq j$$

and

$$\int_\Omega u_i^2 \sigma(u) d\omega_u = \frac{\kappa_{n-1}}{n} S(K), \qquad |u| = 1$$

for $i, j = 1, \cdots, n$ where the integration is extended over the surface Ω of the solid unit sphere.

PROOF. The supporting function $\sigma(u)$ is given, see [4, pp. 48–49], by

$$\sigma(u) = \frac{1}{2} \int_S |u \cdot \tau| \, dS_\tau, \qquad |\tau| = 1 \tag{16}$$

and

$$S(K) = \frac{1}{\kappa_{n-1}} \int_\Omega \sigma(u) d\omega_u, \qquad |u| = 1. \tag{17}$$

Let $S_2(u)$ be any surface harmonic of degree 2, then by the Funk-Hecke theorem [5, pp. 247–248] we have for unit vectors u and τ

$$\int_\Omega |u \cdot \tau| \, S_2(u) d\omega_u = \frac{\pi^{(n-1)/2}}{\Gamma((n+3)/2)} S_2(\tau).$$

Consequently, by (16),

$$\int_\Omega S_2(u) \sigma(u) d\omega_u = \frac{1}{2} \frac{\pi^{(n-1)/2}}{\Gamma((n+3)/2)} \int_S S_2(\tau) dS_\tau$$

and Theorem 3 follows directly from Theorem 1 and (17).

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