NOTE ON THE TENSOR PRODUCT OF BANACH ALGEBRAS[1]

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0. **Introduction.** In [2] the maximal ideal space of the tensor product of two commutative Banach algebras was studied. One of the results was: Let $A_3 = A_1 \otimes_g A_2$ be the "greatest cross-norm" [3; 4] tensor product of two commutative Banach algebras A_1 and A_2. Let $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{M}_3$ be the corresponding spaces of regular maximal ideals. Then $\mathfrak{M}_3$ and $\mathfrak{M}_1 \times \mathfrak{M}_2$ are "naturally" homeomorphic if the weak* topologies are used in all spaces. In the following, we extend the discussion to the case in which no commutativity is assumed.[2]

1. **Tensor products.** Let A_1 and A_2 be Banach algebras and let C be the complex number system. We consider [3] a subset F of $C^{A_1 \times A_2}$:

$$F = \left\{ f \,\middle|\, f \in C^{A_1 \times A_2},\ f(0, x_2) = f(x_1, 0) = 0,\ \sum_{(x_1, x_2)} |f(x_1, x_2)| \, \|x_1\|_1 \|x_2\|_2 < \infty \right\},$$

where $x_i \in A_i$, $i = 1, 2$. Since each f in F is nonzero on a set that is countable or finite, to each f there corresponds a sequence (finite or infinite) $\{(x_{11}, x_{21}), (x_{12}, x_{22}), \cdots\}$ consisting of just the pairs (x_1, x_2) where $f(x_1, x_2) \neq 0$. Addition of elements of f is defined by addition of functions. Multiplication is defined via a form of convolution: If f, $g \in F$, $f * g = h$ is defined by:

$$h(0, x_2) = h(x_1, 0) = 0, \qquad h(x_1, x_2) = \sum_{h_1 b_1 = x_1;\, a_2 b_2 = x_2} f(a_1, a_2) g(b_1, b_2)$$

if $x_1, x_2 \neq 0$. We note:

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$$\sum_{(x_1,x_2)} |h(x_1, x_2)| \, \|x_1\|_1 \|x_2\|_2$$
$$\leqq \sum_{(a_1,a_2),(b_1,b_2)} |f(a_1, a_2)| \, |g(b_1, b_2)| \, \|a_1\|_1 \|b_1\|_1 \|a_2\|_2 \|b_2\|_2$$
$$= \left(\sum_{(a_1,a_2)} |f(a_1, a_2)| \, \|a_1\|_1 \|a_2\|_2 \right) \left(\sum_{(b_1,b_2)} g(b_1, b_2) \|b_1\|_1 \|b_2\|_2 \right) < \infty .$$

Thus $h \in F$ and the value of $h(x_1, x_2)$ is given by an absolutely convergent series. Relative to the operations defined, F is an algebra over C.

If $\|f\|_3'$ is taken as $\sum_{(x_1,x_2)} |f(x_1, x_2)| \, \|x_1\|_1 \|x_2\|_2$, then an analog of the proof that l_1 is a Banach space shows that F is a Banach space. The relation $\|f * g\|_3' \leqq \|f\|_3' \|g\|_3'$ follows from the computation of the preceding paragraph. Hence F is a Banach algebra.

In F let I be the closed ideal generated by all functions f of one of the following forms (x_1, x_1', x_2, x_2' are arbitrary, α is a complex number):

(i) $f(x_1, x_2) = f(x_1, x_2') = -f(x_1, x_2 + x_2')$, $f = 0$ otherwise;

(ii) $f(x_1, x_2) = f(x_1', x_2) = -f(x_1 + x_1', x_2)$, $f = 0$ otherwise;

(iii) $f(x_1, x_2) = -\alpha f(\alpha x_1, x_2)$, $f = 0$ otherwise;

(iv) $f(x_1, x_2) = -\alpha f(x_1, \alpha x_2)$, $f = 0$ otherwise.

We assert that when F/I is given its quotient space norm it is isometrically isomorphic to A_3, the greatest cross-norm tensor product $A_1 \otimes_g A_2$ of A_1 and A_2. The argument rests on the following statements each of which is simply proved.

(a) In F, the set F_0 of functions f that are nonzero on sets that are at most finite is dense in F.

(b) F_0 is a normed algebra containing the generators of I. Let I_0 be the closed ideal generated in F_0 by the generators of I. Then I_0 is a linear space in F_0 and I_0 is dense in I.

(c) For f in F_0, the quotient norm of f/I_0 is precisely the greatest cross-norm of the equivalence class represented by f.

(d) The mapping ϕ_0: $F_0/I_0 \to A_3$ defined by: $\phi_0(f/I_0) =$ *equivalence class represented by* f is an isometry onto a dense subset of A_3. The extension ϕ of ϕ_0 to F/I is an isometry of F/I onto A_3.

In dealing with A_3 (where the norm will be denoted by $\| \cdots \|_3$) we shall often work with representatives of elements in F/I. For any $f \in F$, the element g given by: $g(x_1, x_2) = 1$ if $(x_1, x_2) = (f(x_1', x_2)x_1', x_2)$ and $g = 0$ otherwise, satisfies the equation $f/I = g/I$. Thus, ultimately an element of A_3 will be symbolized by a formal sum $\sum_{n=1}^{\infty} (x_{1n} \otimes x_{2n})$ where $\sum_{n=1}^{\infty} \|x_{1n}\|_1 \|x_{2n}\|_2 < \infty$. Very often we shall omit reference to representatives. For example, we shall say simply "the element

$\sum_{n=1}^{\infty}(x_{1n}\otimes x_{2n})$" when we mean "the element represented by $\sum_{n=1}^{\infty}(x_{1n}\otimes x_{2n})$."[3]

2. $\mathfrak{M}_1\times\mathfrak{M}_2\to\mathfrak{M}_3$. Let (M_1, M_2) be in $\mathfrak{M}_1\times\mathfrak{M}_2$. Define the homomorphism $E_3\colon A_3\to(A_1/M_1)\otimes_g(A_2/M_2)$ by the formula: $E_3(\sum_{n=1}^{\infty}(x_{1n}\otimes x_{2n}))=\sum_{n=1}^{\infty}(E_1(x_{1n})\otimes E_2(x_{2n}))$, where E_1 and E_2 are the canonical epimorphisms $E_i\colon A_i\to A_i/M_i$, $i=1, 2$. Let $\|\cdots\|_{\mathrm{I}}$, $\|\cdots\|_{\mathrm{II}}$ denote the quotient norms in A_1/M_1 and A_2/M_2. Since these algebras are nontrivial so is $(A_1/M_1)\otimes_g(A_2/M_2)$ nontrivial. Thus $E_3^{-1}(0)=M_3$ is a proper ideal in A_3. If u_1 and u_2 are identities modulo M_1 and M_2 then $u=u_1\otimes u_2$ is an identity modulo M_3. Thus M_3 is regular. Since the norm $\|\cdots\|_{\mathrm{III}}$ in $E_3(A_3)$ satisfies

$$\left\|E_3\left(\sum_{n=1}^{\infty}(x_{1n}\otimes x_{2n})\right)\right\|_{\mathrm{III}}\leqq\sum_{n=1}^{\infty}\|E_1(x_{1n})\|_{\mathrm{I}}\|E_2(x_{2n})\|_{\mathrm{II}}$$
$$\leqq\sum_{n=1}^{\infty}\|x_{1n}\|_1\|x_{2n}\|_2,$$

we see that E_3 is norm-decreasing and hence that E_3 is continuous. Hence M_3 is closed.

If M_3 is not maximal, let $M_3\subset N_3$, a regular maximal ideal. Let $F_3\colon A_3\to A_3/N_3$ be the canonical epimorphism and define G_1 and G_2 by the formulas

$$G_1(x_1) = E_3(ux_1),$$
$$G_2(x_2) = E_3(x_2u).$$

Then $G_i(A_i)\subset A_3/M_3$, $i=1, 2$. We prove now

LEMMA 1. *A regular ideal I_3 in A_3 is a right and left A_1- and A_2-ideal.*

PROOF. Let u be an identity modulo I_3 in A_3, $x_1\in A_1$, $v\in I_3$. Then $ux_1v-x_1v\in I_3$ and thus $ux_1v/I_3-x_1v/I_3=0$. But $ux_1v/I_3=(ux_1/I_3)(v/I_3)=0$. Thus $x_1v\in I_3$. Similarly $vx_1\in I_3$, and x_2v, $vx_2\in I_3$ for $x_2\in A_2$.

From Lemma 1 we see that $G_2(x_2y_2)=E_3(x_2y_2u)=E_3(x_2(m_3+uy_2u))=E_3(x_2uy_2u)=G_2(x_2)G_2(y_2)$ (where $m_3\in M_3$). Note also: $\|G_2(x_1)\|_{\mathrm{III}}=\|E_3(ux_1)\|_{\mathrm{III}}\leqq\|ux_1\|_3\leqq k\|x_1\|_1$. It is clear now that G_2 (and similarly G_1) is a continuous homomorphism.

If $(x_1, x_2)\in M_1\times M_2$ then $ux_1=(u_1x_1\otimes u_2)=w$ is such that $E_3(w)=0$, i.e., $G_1(x_1)=0$. Thus $M_1\subset G_1^{-1}(0)$. Similarly, $M_2\subset G_2^{-1}(0)$. Hence either $M_i=G_i^{-1}(0)$ or $G_i(A_i)=0$, $i=1, 2$. Since $G_i(u_i)\neq 0$, we see

[3] A_3 may be regarded as a left and right A_1- and A_2-module, e.g., $x_1(y_1\otimes y_2)=(x_1y_1\otimes y_2)$, where $x_1, y_1\in A_1$, $y_2\in A_2$.

$G_i^{-1}(0)=M_i$. If H_i are engendered by F_3 as G_i are engendered by E_3 we find [2] $M_i=H_i^{-1}(0)$, $i=1, 2$. Following the argument in [2], we obtain $H_i=\alpha_i G_i=\beta_i E_i$ where α_i, β_i are isometric C-automorphisms of A_i/M_i. Finally $E_3=(\beta_1\otimes\beta_2)^{-1}F_3$, whence $E_3(A_3)$ is simple and M_3 is maximal.

We remark that E_3 is an epimorphism. Indeed, if

$$\sum_{n=1}^{\infty}(E_1(x_{1n})\otimes E_2(x_{2n}))=w\in(A_1/M_1)\otimes_g(A_2/M_2),$$

we can choose $(y_{1n}, y_{2n})\in A_1\times A_2$ so that $E_i(y_{in})=E_i(x_{in})$, $i=1, 2$, $n=1, 2, \cdots$, and such that $\|y_{1n}\|_1<\|E_1(x_{1n})\|_{\mathrm{I}}+2^{-n}$, $\|y_{2n}\|_2<\|E_2(x_{2n})\|_{\mathrm{II}}+2^{-n}$. (We may and do assume $\|E_1(x_{1n})\|_{\mathrm{I}}=\|E_2(x_{2n})\|_{\mathrm{II}}$. For, in each nontrivial term we may use the equivalent representative $\alpha E_1(x_{1n})\otimes\alpha^{-1}E_2(x_{2n})$ where $\alpha=+[(\|E_1(x_{1n})\|_1)(\|E_2(x_{2n})\|_{\mathrm{II}})^{-1}]^{1/2}$.) Then if $z=\sum_{n=1}^{\infty}(y_{1n}\otimes y_{2n})$ we see $E_3(z)=w$.

We note (for later use) that $G_1(A_1)$ and $G_2(A_2)$ are commuting subalgebras of A_3/M_3. For

$$\begin{aligned}G_1(x_1)G_2(x_2) &= E_3(ux_1)E_3(x_2u) = E_3(u_1x_1\otimes u_2)E_3(u_1\otimes x_2u_2)\\ &= (E_1(u_1x_1)\otimes E_2(u_2))\cdot(E_1(u_1)\otimes E_2(x_2u_2)) = E_1(x_1)\otimes E_2(x_2)\\ &= (E_1(u_1)\otimes E_2(x_2u_2))\cdot(E_1(u_1x_1)\otimes E_2(u_2)) = E_3(x_2u)E_3(ux_1)\\ &= G_2(x_2)G_1(x_1).\end{aligned}$$

Hence [1] there is a homomorphism $T: G_1(A_1)\otimes_g G_2(A_2)\to A_3/M_3$ given by $T(\sum_{n=1}^{\infty}(G_1(x_{1n})\otimes G_2(x_{2n})))=\sum_{n=1}^{\infty}G_1(x_{1n})G_2(x_{2n})$. Since $T(G_1(u_1)\otimes G_2(u_2))=$ *the identity of* A_3/M_3, and since $G_1(A_1)\otimes_g G_2(A_2)$ is simple we see $T^{-1}(0)=0$, i.e., T is a monomorphism. From our definitions it now follows that T is an isomorphism.

Similarly, if γ_i are isometric isomorphisms $\gamma_i: G_i(A_i)\to B_i$, $i=1, 2$, there is an isomorphism $T': B_1\otimes_g B_2\to A_3/M_3$ given by

$$T'\left(\sum_{n=1}^{\infty}(y_{1n}\otimes y_{2n})\right)=T\left(\sum_{n=1}^{\infty}(\gamma_1^{-1}(y_{1n})\otimes\gamma_2^{-1}(y_{2n}))\right).$$

Finally, the association between pairs (M_1, M_2) and ideals M_3 is 1-1. For if $(M_1, M_2)\neq(M_1', M_2')$, e.g., if $M_1\neq M_1'$, choose $x_1\in M_1\backslash M_1'$ and $x_2\in M_2\backslash M_2'$ if $M_2\neq M_2'$ (otherwise choose $x_2\notin M_2$). Then $E_3(x_1\otimes x_2)=0$, and $E_3'(x_1\otimes x_2)\neq 0$, i.e., (M_1, M_2) and (M_1', M_2') engender distinct epimorphisms E_3 and E_3'.

3. $\mathfrak{M}_3\to\mathfrak{M}_1\times\mathfrak{M}_2$. Consider the algebras A_{ie_i} obtained by adjoining (if necessary) identities e_i to A_i, $i=1, 2$. It is to be noted that $A_4=A_{1e_1}\otimes_g A_{2e_2}$ is in general not the same as A_{3e_3}, the result of

adjoining (if necessary) an identity e_3 to A_3. A typical element z of A_4 may be represented by $\alpha(e_1\otimes e_2)+e_1\otimes x_2+x_1\otimes e_2+\sum_{n=1}^{\infty}(x_{1n}\otimes x_{2n})$ where $x_1, x_{1n}\in A_1$, $x_2, x_{2n}\in A_2$, $\alpha\in C$, $\sum_{n=1}^{\infty}\|x_{1n}\|_1\|x_{2n}\|_2<\infty$. There is an obvious isometric isomorphic copy of A_3 in A_4. We denote the copy of A_3 by A_3.

LEMMA 2. *A_3 is an ideal in A_4.*

PROOF. Let z be as above, and let $w=\sum_{n=1}^{\infty}(w_{1n}\otimes w_{2n})\in A_3$. Then $zw=\alpha w+wx_2+x_1w+vw$ where $v=\sum_{n=1}^{\infty}(x_{1n}\otimes x_{2n})$. Since all four summands of zw are in A_3 the result follows.

A similar argument employing Lemma 1 shows the following.

LEMMA 3. *A closed ideal I_3 of A_3 is a closed ideal in A_4.*

LEMMA 4. *Let M_3 be a regular maximal ideal of A_3, u an identity (in A_3)* modulo *M_3: uA_3-A_3, $A_3u-A_3\subset M_3$. In A_4 let $M_4^0=\{z\mid uz, zu\in M_3\}$, $A_4^0=[x_1\otimes e_2+e_1\otimes x_2]\oplus A_3$. Then M_4^0 is a (regular) ideal in A_4, u is an identity (in A_4)* modulo *M_4^0 and $M_4^0\not\subset A_4^0$.*

PROOF. For $z\in M_4^0$, $w\in A_4$, we see in succession: uz, uzw, $uzwu$, $u(zwu)-zwu$, $zwu\in M_3$. Hence M_4^0 is a right ideal and by a similar proof is a left ideal.

Writing $e_4=e_1\otimes e_2$, we see $u(u-e_4)=u^2-u\in M_3$, whence $u-e_4\in M_4^0\backslash A_4^0$.

Since (Lemma 2) A_3 is an ideal in A_4, we find that if $z\in A_4$ then uz, $zu\in A_3$ and $(uz-z)u=u(zu)-zu$, $u(uz-z)=u(uz)-uz\in M_3$, whence $uz-z\in M_4^0$.

We now consider several cases:

Case I. Assume A_1 and A_2 have identities. Define mappings F_i: $A_i\to A_3/M_3$, $i=1, 2$ by the (symbolically given) formulas $F_1(A_1)=(A_1\otimes e_2)/M_3$, $F_2(A_2)=(e_1\otimes A_2)/M_3$. Then $F_i^{-1}(0)$ are regular ideals N_i of A_i, $i=1, 2$.

We find the pair associated with M_3 by treating several subcases.

Case 1. *N_1 and N_2 are both maximal.* Then as in §2, let them engender an $N_3\in\mathfrak{M}_3$ and let G_3 be the canonical epimorphism G_3: $A_3\to(A_1/N_1)\otimes_g(A_2/N_2)$. We shall show $N_3\subset M_3$ (whence we shall have shown $N_3=M_3$ since both are maximal ideals). Thus let $z=\sum_{n=1}^{\infty}(x_{1n}\otimes x_{2n})\in A_3$ and assume $G_3(z)=0$. Then

$$\sum_{n=1}^{\infty}(F_1(x_{1n})\otimes F_2(x_{2n}))=0.$$

$F_1(A_1)$ and $F_2(A_2)$ are commuting subalgebras of A_3/M_3 and thus T: $(A_1/N_1)\otimes_g(A_2/N_2)\to A_3/M_3$, given by $T(\sum_{n=1}^{\infty}(F_1(x_{1n})\otimes F_2(x_{2n})))$

$= \sum_{n=1}^{\infty} F_1(x_{1n})F_2(x_{2n})$, is a homomorphism. (Since $T(F_1(u_1)\otimes F_2(u_2)) =$ *the identity* of A_3/M_3, and since $F_1(A_1)\otimes_g F_2(A_2)$ is simple, T is a monomorphism.) Hence

$$F_3(z) = \sum_{n=1}^{\infty} (F_1(x_{1n})F_2(x_{2n})) = T\left(\sum_{n=1}^{\infty} (F_1(x_{1n}) \otimes F_2(x_{2n}))\right) = T(0) = 0.$$

Thus $N_3 \subset M_3$.

Case 2. *N_1 is not maximal; N_2 is maximal.* Let $N_1 \subset M_1 \in \mathfrak{M}_1$. Then M_1 and N_2 engender an $M_3' \in \mathfrak{M}_3$. We shall show $M_3' \subset M_3$ (whence, again, we shall have shown $M_3' = M_3$). This time let G_3' be the canonical epimorphism $G_3' : A_3 \to (A_1/M_1)\otimes_g (A_2/N_2)$, and let G_1', G_2' arise from G_3' according to the procedure in §2. Then $\alpha_2 G_2' = F_2$, where α_2 is an isometric C-automorphism of $F_2(A_2)$. As indicated in §2, there is an isomorphism $T: G_1'(A_1)\otimes_g G_2'(A_2) \to A_3/M_3'$. If $L: A_1/N_1 \to (A_1/N_1)/(F_1(M_1))$ is the canonical epimorphism, then by virtue of the "second isomorphism theorem" $LF_1 = \alpha_1 G_1'$, where α_1 is an isometric C-automorphism of $G_1'(A_1)$. If $G_3(z) = 0$, then $\sum_{n=1}^{\infty} (E_1(x_{1n})\otimes F_2(x_{2n})) = 0$, ($E_1$: $A_1 \to A_1/M_1$). Via C-automorphisms we can conclude $\sum_{n=1}^{\infty} (G_1'(x_{1n})\otimes G_2'(x_{2n})) = 0$ and then that

$$(\alpha_1 \otimes \alpha_2)\left(\sum_{n=1}^{\infty} (G_1'(x_{1n}) \otimes G_2'(x_{2n}))\right) = \sum_{n=1}^{\infty} (\alpha_1 G_1'(x_{1n}) \otimes \alpha_2 G_2'(x_{2n}))$$

$$= \sum_{n=1}^{\infty} (LF_1(x_{1n}) \otimes F_2(x_{2n})) = 0.$$

Remembering that α_1 and α_2 are automorphisms and applying T we find $\sum_{n=1}^{\infty} LF_1(x_{1n})F_2(x_{2n}) \equiv L_3(\sum_{n=1}^{\infty} F_1(x_{1n})F_2(x_{2n})) = L_3F_3(z) = 0$ (where L_3: $A_3/M_3 \to A_3/M_3'$ is defined by the initial identity). Since $L_3(F_1(u_1)F_2(u_2)) \neq 0$ and since A_3/M_3 is simple we see $F_3(z) = 0$, and so $M_3' \subset M_3$.

Case 3. *Both N_1 and N_2 are not maximal.* Argue *mutatis mutandis* as in Case 2.

Case II. Not both A_1 and A_2 have identities. Let M_4 be a regular maximal ideal in A_4 such that (Lemma 4) $A_4^0 \not\supset M_4 \supset M_3$. Let M_4 engender M_{ie_i}, regular maximal ideals of A_{ie_i}, $i = 1, 2$ (Case I). If M_{1e_1}, say, is A_1 then clearly $A_3/M_4 = 0$, whence $u/M_4 = 0$, a contradiction. Hence $M_{ie_i} \not\subset A_i$, and the regular maximal ideals we seek are $M_i \equiv M_{ie_i} \cap A_i$, $i = 1, 2$.

In every case we get a pair (M_1, M_2) in $\mathfrak{M}_1 \times \mathfrak{M}_2$ and this pair in turn engenders M_3. Since the association $\mathfrak{M}_1 \times \mathfrak{M}_2 \to \mathfrak{M}_3$ is 1-1, (M_1, M_2) is uniquely determined by M_3: $(M_1, M_2) = \bar{t}(M_3)$. We have shown $\bar{t}$ is 1-1 and that $\bar{t}(\mathfrak{M}_3) = \mathfrak{M}_1 \times \mathfrak{M}_2$.

THEOREM. *When hk-topologies are used throughout, $\bar{\imath}$ is continuous but not generally bicontinuous.*

PROOF. We shall show that $\bar{\imath}^{-1}$ is closed. Let F_1 be closed in $\mathfrak{M}_1$. Then $F_3 = F_1 \times \mathfrak{M}_2$ is closed in $\mathfrak{M}_1 \times \mathfrak{M}_2$. Let $K_3 = \bar{\imath}^{-1}(F_3)$. We show that K_3 is closed. To this end let $M_3^0 \supset k(K_3)$ and assume $M_3^0 \notin K_3$. Thus if $\bar{\imath}(M_3^0) = (M_1^0, M_2^0)$, then $M_1^0 \notin F_1$. Hence $M_1^0 \not\supset k(F_1)$ and there is an x in $k(F_1)$ such that $x/M_1^0 \neq 0$. But if y is an arbitrary element of A_2, then $z = x \otimes y$ in A_3 is actually in $k(K_3)$. For if $M_3 = \bar{\imath}^{-1}(M_1, M_2)$ is in K_3, then M_1 is in F_1 and hence $x/M_1 = 0$, whence $z/M_3 = (x/M_1) \otimes (y/M_2) = 0$. Therefore z is in M_3, and since M_3 is arbitrary in K_3, z is in $k(K_3)$. Thus any such z is in M_3^0. But if we choose y so that $y/M_2^0 \neq 0$, then $z/M_3^0 \neq 0$. This contradiction shows that K_3 is closed. Similarly, if F_2 is closed in $\mathfrak{M}_2$ then $\bar{\imath}^{-1}(\mathfrak{M}_1 \times F_2)$ is closed.

Since any closed set in $\mathfrak{M}_1 \times \mathfrak{M}_2$ is of the form

$$\bigcap_r \left[(F_1^r \times \mathfrak{M}_2) \cup (\mathfrak{M}_1 \times F_2^r)\right]$$

where F_i^r is closed in $\mathfrak{M}_i$, $i = 1, 2$, it follows that $\bar{\imath}^{-1}$ is closed and thus that $\bar{\imath}$ is continuous.

We are indebted to John Lindberg for suggesting the following example showing that $\bar{\imath}^{-1}$ need not be continuous.

EXAMPLE. Let A_1 be the commutative Banach algebra of functions analytic in the interior of the unit disc D_z and continuous on the entire unit disc D_z. The maximal ideal space of A_1 may be identified with D_z although the *hk*-topology constitutes a genuine weakening of the usual topology of D_z. In fact, a *hk*-closed set F in D_z has only countably many points in the (usual) interior of D_z unless F is the whole of D_z. Let A_2 be $C(D_w)$, i.e., the algebra of all continuous functions on D_w. Then the maximal ideal space of A_2 in the *hk*-topology is homeomorphic with D_w in its usual topology.

In A_3 let v be the element $z \otimes 1 - 1 \otimes w$ (z and w here are the usual complex variables). Then in $\mathfrak{M}_1 \times \mathfrak{M}_2$ the set $K = \{(z, w) \mid z = w\}$ is the hull of v, and so $\bar{\imath}^{-1}(K)$ is closed in the *hk*-topology of $\mathfrak{M}_3$. Thus $U = \mathfrak{M}_3 \backslash \bar{\imath}^{-1}(K)$ is open. We shall show that $\bar{\imath}(U)$ is not open in $\mathfrak{M}_1 \times \mathfrak{M}_2$.

If $\bar{\imath}(U)$ were open, there would be open sets U_i in $\mathfrak{M}_i$ such that $U_1 \times U_2 \subset \bar{\imath}(U)$, and U_2 could be taken as an ordinary open circle in the interior of D_w. Since $F_1 = \mathfrak{M}_1 \backslash U_1$ is *hk*-closed in D_z, F_1 has at most countably many points in the ordinary interior of D_z. Let U_2^* be the circle in D_z that consists of the same set of complex numbers as the set comprising U_2. Then $U_2^* \backslash F_1 \neq \emptyset$ since F_1 meets the ordinary interior of D_z in at most countably many points. But then $U_1 \times U_2 \supset (U_2^* \backslash F_1) \times U_2$. Clearly, any point z in $U_2^* \backslash F_1$ corresponds to a point

w in U_2 such that $z=w$, i.e., $(U_2^*\backslash F_1)\times U_2$ meets K and thus $U_1\times U_2$ cannot lie in the complement $\bar{t}(U)$ of K.

REMARK. In the last paragraph of p. 302 and the first two lines of p. 303 [2] the proof given is incorrect since the u_i chosen there vary with the M_i. The following amended proof should be substituted.

Let M_{03} be in $\mathfrak{M}_3$ and if $\bar{t}(M_{03})=(M_{01}, M_{02})$, let $N(M_{0i})$ in $\mathfrak{M}_i$ be of the form:

$$N(M_{0i}) = \{M_i \mid \mid \overset{+}{a}_{ji}(M_i) - \overset{+}{a}_{ji}(M_{0i})\mid < r_i, j = 1, 2, \cdots, J_i\}, \quad i = 1, 2.$$

To prove the continuity of $\bar{t}$ it suffices to find an $N(M_{03})$ such that $\bar{t}(N(M_{03}))\subset N(M_{01})\times N(M_{02})$. To this end let u_{0i} be identities modulo M_{0i}, $i=1, 2$ and let

$$\begin{aligned} N(M_{03}) = \{M_3 \mid \mid (\overset{k}{a}_{j1}\otimes u_{02})^+(M_3) - (\overset{k}{a}_{j1}\otimes u_{02})^+(M_{03})\mid &< s_k, \\ k = 1, 2, j = 1, 2, \cdots, J_1; \\ \mid (u_{01}\otimes \overset{m}{a}_{j2})^+(M_3) - (u_{01}\otimes \overset{m}{a}_{j2})^+(M_{03})\mid &< v_m, \\ m = 1, 2, j = 1, 2, \cdots, J_2; \\ \mid (u_{01}\otimes u_{02})^+(M_3) - 1\mid &< w\}. \end{aligned}$$

We treat two cases: (1) For all (j, i) $\overset{+}{a}_{ji}(M_{0i})\neq 0$. Then for appropriate choices of the s_k, v_m and w, we can assure that

$$\left|\frac{(\overset{2}{a}_{j1}\otimes u_{02})^+(M_{03})}{(a_{j1}\otimes u_{02})^+(M_{03})} - \frac{(\overset{2}{a}_{j1}\otimes u_{02})^+(M_3)}{(a_{j1}\otimes u_{02})^+(M_3)}\right| \equiv \mid \overset{+}{a}_{j1}(M_{01}) - \overset{+}{a}_{j1}(M_1)\mid < r_1$$

and similarly that $\mid \overset{+}{a}_{j2}(M_{02}) - \overset{+}{a}_{j2}(M_2)\mid < r_2$, where $\bar{t}(M_3)=(M_1, M_2)$. In this case then, $\bar{t}(N(M_{03}))\subset N(M_{01})\times N(M_{02})$. (2) Some $\overset{+}{a}_{ji}(M_{0i})=0$. For the (j, i) combinations for which $\overset{+}{a}_{ji}(M_{0i})\neq 0$, s_k, v_m and w are chosen as in (1). For the other (j, i) combinations, since $\mid \overset{+}{u}_{0i}(M_i)\mid$ is bounded, we see that for suitable w and M_3 in $N(M_{03})$ $\mid \overset{+}{u}_{0i}(M_i)\mid$ is bounded away from 0 and then for suitable s_k and v_m $\mid \overset{+}{a}_{ji}(M_i)\mid$ is small. The result follows upon a final revised choice of s_k, v_m and w.

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