A PROPERTY OF THE REAL LINE EQUIVALENT TO THE CONTINUUM HYPOTHESIS

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1. **Introduction.** Edwin Hewitt has asked if the real line R is a normal topological space (see below for the definition of this term) when it is given the topology $\mathfrak{J}$ which is defined in the following way. Fix a Hamel basis H for R over the rational numbers. For $x \in R$ and $a \in H$, let x_a denote the ath coordinate of x in its expansion with respect to H. Now, for each countable subset $K \subset H$ and each ϵ such that $0 < \epsilon \leqq \infty$, define

$$V(K, \epsilon) = \{x \in R: |x| < \epsilon \text{ and } x_a = 0 \text{ for } a \in K\}.$$

Then $\mathfrak{J}$ is the group topology on R that has the collection of all possible $V(K, \epsilon)$ as a basis of open sets at 0. The space $(R, \mathfrak{J})$ is obviously completely regular, since it is a topological group. Theorem 1 answers Hewitt's question. Before stating the theorem, let us recall that a topological space X is *normal* if any two disjoint closed sets have disjoint neighborhoods.

THEOREM 1. *The real line R under the topology $\mathfrak{J}$ defined above is normal if and only if the continuum hypothesis is true.*

2. **Proof of sufficiency.** First a lemma is stated, next it is pointed out how the lemma implies the desired result, and finally a proof for the lemma is given. For use in stating the lemma, recall that a collection $\mathfrak{U}$ of disjoint subsets of a topological space X is *discrete* if $x \in (\bigcup \mathfrak{U})^-$ implies that $x \in U^-$ for some $U \in \mathfrak{U}$. (The bar indicates closure in X.) As for notation, we adopt the convention that i and j run over all integers $\geqq 0$, and n runs over all integers $\geqq 1$.

LEMMA 1. *If the continuum hypothesis is true, then there exist a countable number of subsets R_i of R and a countable number of collections $\mathfrak{V}_i$ such that*

(1) $R = \bigcup_i R_i$;
(2) *for $x \in R_i$, there is a unique $V_{i,x} \in \mathfrak{V}_i$ with $x \in V_{i,x}$;*
(3) *for $x \in R_i$, $y \in R_i$, $V_{i,x} \cap V_{i,y} = \emptyset$ if $x \neq y$;*
(4) *for each i, $\mathfrak{V}_i$ is a discrete open collection.*

Let us show how Lemma 1 is applied. From [2, 5.32] it will be seen that it is sufficient to prove that $(R, \mathfrak{J})$ is paracompact. This latter term may be defined as follows. X is *paracompact* if each open

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cover of X has an open refinement consisting of a countable union of discrete collections. (See [2, 5.28].) Now, if $\mathcal{U}$ is an open cover for R, choose for each $x \in R$ a $U_x \in \mathcal{U}$ with $x \in U_x$. Then define $\mathcal{W}_i = \{V_{i,x} \cap U_x : x \in R_i\}$. It is easy to verify that $\mathcal{W} = \bigcup_i \mathcal{W}_i$ has the desired properties.

Proof of Lemma 1. In the way of notation, define for each integer $i \geqq 0$

$$T_i = \{x \in R: x_a \neq 0 \text{ for exactly } i \text{ of the } a \in H\}.$$

It will be proved that there are a countable number of subsets $T_{i,n}$ of T_i and a countable number of collections $\mathcal{W}_{i,n}$ such that

(1) $T_i = \bigcup_n T_{i,n}$;

(2) for $x \in T_{i,n}$, there is a unique $W_n(x) \in \mathcal{W}_{i,n}$ with $x \in W_n(x)$;

(3) for $x \in T_{i,n}$ and $y \in T_{i,n}$, $W_n(x) \cap W_n(y) = \emptyset$ if $x \neq y$;

(4) for each n, $\mathcal{W}_{i,n}$ is a discrete open collection.

(Note that in (4) we mean discrete and open with respect to $(R, \mathfrak{J})$.) It is not difficult to verify that Lemma 1 is a consequence of this.

We begin by using the continuum hypothesis to number the Hamel basis H from 1 to (but not including) Ω, where Ω is the first uncountable ordinal. For $1 \leqq \alpha < \Omega$, $a(\alpha)$ will denote the αth $a \in H$. Also, for $x \neq 0$, let δ_x be the largest ordinal α such that $x_{a(\alpha)} \neq 0$. Now introduce a relation on R by defining $y \prec x$ to mean that $x \neq y$, but $y_a \neq x_a$ only if $x_a \neq 0$ and $y_a = 0$. Note that each x has a finite number of predecessors.

For $x \neq 0$, define

$$W(x) = \{z \in R: z_{a(\alpha)} = x_{a(\alpha)} \text{ for } \alpha \leqq \delta_x\};$$

and for each n, define

$$W_n(x) = \{z \in W(x): |z - y| > 1/n \text{ for all } y \prec x\}.$$

When $i > 0$, let $\mathcal{W}_{i,n} = \{W_n(x): x \in T_i\}$, and set $T_{i,n} = [\bigcup \mathcal{W}_{i,n}] \cap T_i$. When $i = 0$, let $T_{0,n} = \{0\}$ for all n, and let $\mathcal{W}_{0,n} = \{R\}$.

Let us check that these sets satisfy (1)–(4) above. This is obvious when $i = 0$; hence we suppose that $i > 0$. Clearly, for $x \in T_i$, one can choose n so large that $|x - y| > 1/n$ for all $y \prec x$. Hence, for this n, $x \in W_n(x) \in \mathcal{W}_{i,n}$, and (1) follows from this. Moreover, if x and y are both in T_i, $W(x) \cap W(y) \neq \emptyset$ only if $x = y$; hence, for x and y in $T_{i,n}$, $W_n(x) \cap W_n(y) \neq \emptyset$ only if $x = y$. Property (3) is a consequence of this, and (2) follows from this and the definition of $T_{i,n}$.

In proving (4), let us first verify that $\mathcal{W}_{i,n}$ is an open collection. Note that each $W(x)$ is open, since $W(x) = x + V(K, \infty)$, where $K = \{a(\alpha): \alpha \leqq \delta_x\}$. Also, since each member of the usual topology for R is in $\mathfrak{J}$, $U = \{z \in R: |z - y| > 1/n \text{ for all } y \prec x\}$ is in $\mathfrak{J}$. Hence $W_n(x) = W(x) \cap U$ is open.

Finally, to show that $\mathcal{W}_{i,n}$ is a discrete collection, suppose $z\in(\bigcup\mathcal{W}_{i,n})^-$. Now $z\in T_p$ for some integer p. If $p\geqq i$, then let y be defined by choosing $y_a=z_a$ except for the $p-i$ largest a for which $z_a\neq 0$ (where "largest" refers to the ordering of H); for the excepted a, let $y_a=0$. For this y, $z\in W(y)$ and $y\in T_i$; hence $W(y)$ is a neighborhood of z which meets only one member of $\mathcal{W}_{i,n}$, namely $W_n(y)$.

All that remains is the case $p<i$. It will be proved that, for z and p as above, this case cannot occur. Suppose $z\in T_p$, $p<i$, and let

$$U = \{y\in W(z): |y-z| < 1/n\}.$$

We will prove that U is a neighborhood of z which does not meet $\bigcup\mathcal{W}_{i,n}$. Suppose to the contrary that $U\cap W_n(x)\neq\emptyset$ for some $x\in T_i$, and choose a $y\in U\cap W_n(x)$. Note that $z<x$, since $U\cap W_n(x)\neq\emptyset$ implies that $W(z)\cap W(x)\neq\emptyset$, and the latter implies that $x_{a(\alpha)}=z_{a(\alpha)}$ for all $\alpha\leqq\delta_z$. This leads to a contradiction because $y\in W_n(x)$ and $z<x$ imply $|y-z|>1/n$; but $y\in U$ implies $|y-z|<1/n$. (Note that U is open for the same reason that each $W_n(x)$ is open.)

COROLLARY. *Assume the continuum hypothesis. Then every subspace of* $(R, \mathfrak{J})$ *is paracompact, and hence normal. Also, every subset of* $(R, \mathfrak{J})$ *is an* F_σ. *(An* F_σ *is a subset which is the union of a countable number of closed subsets.)*

PROOF. Note that Lemma 1 remains true if R is replaced by a subspace A of R. Hence A is paracompact as above. Also, observe that the properties of $\mathcal{V}_i$ imply that each subset of R_i is closed. Hence $A=\bigcup_i (R_i\cap A)$ with each $R_i\cap A$ closed in $(R, \mathfrak{J})$.

REMARK. The last assertion of the corollary can be derived without the continuum hypothesis by an argument similar to that appearing in the first paragraph of the proof of Lemma 2 in the next section.

3. **Proof of necessity.** Throughout this section it will be assumed that $\aleph_1<2^{\aleph_0}$. We will suppose that $(R, \mathfrak{J})$ is normal and argue for a contradiction. This will follow a sequence of lemmas.

Let i be the least integer such that T_i is of second category in R—where, from here on, R will denote the real numbers under their usual topology.

LEMMA 2. $X=\bigcup_j T_{i+j}$ *is a normal subspace of* $(R, \mathfrak{J})$.

PROOF. It will be proved that X is an F_σ in $(R, \mathfrak{J})$. It is sufficient to show this, since it is well known that an F_σ in a normal space is normal. Define, for integers j and n, $T_{j,n}=\{x\in T_j: |x-y|>1/n$ for all $y<x\}$. (The $T_{j,n}$ of the last section could have been defined this way.) One may easily verify that $T_j=\bigcup_n T_{j,n}$. Hence it will suffice to prove that each $T_{j,n}$ is closed in $(R, \mathfrak{J})$.

To accomplish the latter, first suppose that $y\in T_k\backslash T_{j,n}$ $(k\geqq j)$ and $K=\{a\in H: y_a\neq 0\}$. Then $y+V(K,\infty)$ is an open neighborhood of y which does not meet $T_{j,n}$. Now assume that $y\in T_k$ $(k<j)$ and that K is as above. Then $y+V(K,1/n)=V$ does not meet $T_{j,n}$. In fact, if $x\in T_j\cap V$ then $y_b=x_b$ for all b such that $y_b\neq 0$. Hence $y\prec x$, but $|x-y|<1/n$. It follows that $x\in' T_{j,n}$.

For convenience, let us suppose that, whenever a rational number is written in the form m/n, we have m and n relatively prime. Define A to be the set of $x\in T_i$ such that, for any a, the m in the expression $x_a=m/n$ is even. Let $B=T_i\backslash A$. It follows by an argument similar to part of the proof of Lemma 2 that each subset of T_i is closed in X. Hence A and B are both closed in X. By the normality of X, one may choose disjoint U and V, each open subsets of X, such that $A\subset U$ and $B\subset V$.

By the definition of the topology $\mathfrak{I}$, one may choose, for each $x\in A\cup B$, a countable $K(x)\subset H$ and $\epsilon(x)>0$ such that $V(x) =[x+V(K(x),\epsilon(x))]\cap X$ is contained in U if $x\in A$, or in V if $x\in B$. Using this notation we will now describe some of the properties of A and B.

LEMMA 3. *A is of second category in R.*

PROOF. Either A or B is of second category in R, since $A\cup B=T_i$. For $x\in B$, $2^n x\in A$ if n is large. Let $B_n=\{x\in B: 2^n x\in A\}$. If B is of second category in R, then so is B_{n_0} for some n_0. Hence A, which contains $2^{n_0}B_{n_0}$, is of second category in R.

For $S\subset R$ and $K\subset H$, let $S(K)=\{x\in S: x_a=0 \text{ when } a\in K\}$.

LEMMA 4. *If $S\subset T_i$ is of second category in R, then $S(K)$ is of second category in R for each countable subset K of H.*

PROOF. Let gK denote the group generated by K. Suppose $x\in gK\backslash\{0\}$. Let $S_x=(S-x)(K)$. Since $S_x\subset\{T_r: 0\leqq r<i\}$, S_x is of first category in R by the minimality of i. Hence S_x+x is of first category in R. Since K is countable, $T=\bigcup\{S_x+x: x\in gK\backslash\{0\}\}$ is of first category in R. Consequently, $S(K)=S\backslash T$ is of second category in R.

For each set S let $|S|$ denote the cardinality of S. Let us say that an interval I (all intervals which occur here will be open) is *$\aleph_2$-filled* with a subset S of R if, for each $K\subset H$, $|K|<\aleph_2$, one has $|S(K)\cap I| \geqq\aleph_2$.

LEMMA 5. *If I is $\aleph_2$-filled with a set S which is the union of a countable collection of S_n, then for some n_0 there are arbitrarily small intervals contained in I which are $\aleph_2$-filled with S_{n_0}.*

PROOF. Suppose to the contrary that, for each n, there is some $\theta(n)>0$ such that, given an interval J of length less than $\theta(n)$, one may choose $K_J \subset H$ with the property that $|J \cap S_n(K_J)| < \aleph_2$ and $|K_J| < \aleph_2$. For a fixed n, let $J_1, J_2, \cdots$ be a countable cover of I by intervals of length less than $\theta(n)$. Define K_n as $\bigcup_j K_{J_j}$, and K as $\bigcup_n K_n$. Then $S_n(K_n) \subset \bigcap_j S_n(K_{J_j})$; hence $S_n(K_n) \cap I \subset \bigcup_j (S_n(K_{J_j}) \cap J_j)$, and $|S_n(K_n) \cap I| < \aleph_2$. Also, $S(K) \subset \bigcup_n S_n(K_n)$; hence $|S(K) \cap I| \leqq |\bigcup_n (S_n(K_n) \cap I)| < \aleph_2$. Since $|K| < \aleph_2$, this contradicts the assumption that I is $\aleph_2$-filled with S.

LEMMA 6. *Each interval I is $\aleph_2$-filled with B.*

PROOF. Suppose $|K| < \aleph_2$. Then $|B(K)| > \aleph_1$ as one may easily verify. Let $x \in R \setminus \{0\}$ be chosen such that each interval containing x also contains $\aleph_2$ elements of $B(K)$. Choose a rational number r of the form $m/2^n$ $(n>0)$ such that $rx \in I$. Note that $rB(K) \subset B(K)$, since m and 2^n are relatively prime. Hence $|B(K) \cap I| \geqq |rB(K) \cap I| \geqq \aleph_2$.

For the remainder of this section, let n be an integer, and let I be an interval, such that n and I have the properties of the next lemma. (For the statement of Lemma 7, recall the definition of $\epsilon(x)$ in the expression of $V(x)$.)

LEMMA 7. *There are an integer n and an interval I such that* (1) *if $A_n = \{x \in A : \epsilon(x) > 1/n\}$ then $A_n \cap I$ is of second category in R,* (2) *I is $\aleph_2$-filled with B_n where $B_n = \{x \in B : \epsilon(x) > 1/n\}$, and* (3) *$I$ is of length less than $1/n$.*

PROOF. For (1) choose an integer p such that A_p is of second category in R. It is not difficult to show that, for some interval J, $A_p \cap L$ if of second category in R for each subinterval L of J. Now J is $\aleph_2$-filled with B by Lemma 6, and by Lemma 5 there is a q such that there are arbitrarily small subintervals of J which are $\aleph_2$-filled with B_q. Let $n = \max(p, q)$. Choose I to be a subinterval of J such that (3) is satisfied for I and (2) is satisfied for B_n and I. It follows that (1), (2), and (3) are satisfied for n and I.

This completes the sequence of lemmas, and we now observe the following facts:

(i) *If $x \in (A_n \cup B_n) \cap I$, then $V(x) \cap I = W(x) \cap I$, where* $W(x) = [x + V(K(x), \infty)] \cap X$. For suppose $y \in W(x) \cap I$; then $|y-x| < 1/n < \epsilon(x)$, and $y \in V(x) \cap I$.

(ii) *If $y \in W(x) \cap W(x')$ for $x \in A_n \cap I$ and $x' \in B_n \cap I$, then there is some $y' \in W(x) \cap W(x') \cap I$.* In fact, let $a \in H \setminus (K(x) \cup K(x'))$ be chosen such that $y_a = 0$. Now pick a rational number r such that $y' = y + ra \in I$. It follows that y' has the desired property.

(iii) *For each* $x \in A_n \cap I$ *and each* $x' \in B_n \cap I$, $W(x) \cap W(x') = \varnothing$. In fact, suppose $y \in W(x) \cap W(x')$ for some choice of $x \in A_n \cap I$ and $x' \in B_n \cap I$. Pick $y' \in X$ as in (ii). Then by (i), $y' \in V(x) \cap V(x')$, which contradicts the assumption that $U \cap V = \varnothing$.

This section will be completed by proving that (iii) is false. To accomplish this, choose by induction on α an

$$x(\alpha) \in (I \cap A_n)(\bigcup\{K(x(\beta)) : \beta < \alpha\})$$

for $1 \leqq \alpha < \Omega$ such that

$$(\delta) \qquad (x(\alpha))_a = 0 \text{ if } (x(\beta))_a \neq 0 \text{ for any } \beta < \alpha.$$

This is accomplished by using Lemma 4 and the fact that $A_n \cap I$ is of second category in R. Now pick $x_0 \in B_n \cap I$ such that $(x_0)_a = 0$ if either $a \in \bigcup\{K(x(\alpha)) : 1 \leqq \alpha < \Omega\}$ or $(x(\alpha))_a \neq 0$ for some $1 \leqq \alpha < \Omega$. This is accomplished by using the fact that I is $\aleph_2$-filled with B_n.

We will contradict (iii) by proving that, for a sufficiently large α, $W(x_0) \cap W(x(\alpha)) \neq \varnothing$. In fact, pick α large enough so that $(x(\alpha))_a = 0$ if either $a \in K(x_0)$ or $(x_0)_a \neq 0$. (This is possible because of condition (δ).) One may verify that, for this α, $x_0 + x(\alpha) \in W(x_0) \cap W(x(\alpha))$. This completes the proof.

4. **Remarks.** (a) The first three lemmas of §3 can be either much simplified or omitted when the Hamel basis is of second category in R. However, this is not always so; for instance, a maximal linearly independent subset H of a set which is both of positive measure and of first category in R can be proved to be a Hamel basis. On the other hand, V. L. Klee has pointed out to me that a method due to F. B. Jones [Bull. Amer. Math. Soc. vol. 48 (1942) pp. 115–120] demonstrates the existence of a Hamel basis H' which is of second category in R.

As in §1, define a topology $\mathfrak{J}$ on R using H, and a topology $\mathfrak{J}'$ using H', where H is of first category in R and H' is of second category in R. One might reasonably conjecture that $(R, \mathfrak{J})$ and $(R, \mathfrak{J}')$ are homeomorphic. However, let i be the identity function on R, and let f be defined by extending linearly a fixed one-to-one correspondence between H and H'. It can be proved that f is not continuous and that i is not necessarily continuous, where both are considered as functions from $(R, \mathfrak{J})$ to $(R, \mathfrak{J}')$. Hence, it does not seem that a natural homeomorphism exists, and it is necessary to have a proof that applies to either possibility for H.

(b) There are several other topologies for R whose properties complement those of $\mathfrak{J}$. Define $\mathfrak{J}(\aleph, \epsilon)$ to be the group topology on R

generated by all $V(K, \epsilon)$ with $|K| < \aleph$ and $\epsilon > 0$, and define $\mathfrak{J}(\aleph)$ as the group topology generated by all $V(K, \infty)$ with $|K| < \aleph$. Without any assumptions about the continuum hypothesis one may prove that $\mathfrak{J}(\aleph_0, \epsilon)$ is not normal, that $\mathfrak{J}(2^{\aleph_0}, \epsilon)$ is normal (and paracompact), and that $\mathfrak{J}(\aleph)$ is normal (and paracompact) for any $\aleph$. Moreover, for $\mathfrak{J}(\aleph)$ with $\aleph \leqq \aleph_1$, it may be proved that this topology is Lindelöf by the methods of [1, Proposition 3]. (A space is Lindelöf if each open cover has a countable subcover. See [2, 5.Y] for a proof that Lindelöf implies paracompact.)

(c) Note that it is also true that $2^{\aleph_0} = \aleph_1$ if and only if $(R, \mathfrak{J})$ is paracompact; but a proof of this may be constructed which is not so involved as that for Theorem 1.

References

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