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EXTENDING CHARACTERS ON SEMIGROUPS

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W. W. Comfort has proved [1, Theorem 4.2] a theorem on approximating certain semicharacters on commutative semigroups. He used the structure theory established in [2] and expressed doubt as to the necessity of one of his hypotheses, namely core $S(\chi)\neq\Lambda$. His result suggested the following theorem, which tells us when a character on a subsemigroup of a commutative semigroup G can be extended to a character on G. Because of its technical nature we will not state Comfort's theorem but we will state as a corollary to our theorem a result which implies his theorem directly (with the hypothesis core $S(\chi)\neq\Lambda$ dropped).

A bounded complex-valued function ψ on a semigroup G is called a *semicharacter* of G if $\psi(x)\neq 0$ for some $x\in G$ and $\psi(xy)=\psi(x)\psi(y)$ for all $x, y\in G$. A *character* ψ is a semicharacter for which $|\psi(x)|=1$ for all $x\in G$. We note that it follows from the theorem in [3] that any character can be extended to a semicharacter.

THEOREM. *Let G be a commutative semigroup and let $S\subseteq G$ be a subsemigroup. A character ψ on S can be extended to a character on G if and only if ψ satisfies*:

$$(*)\qquad a, b \in S,\ x \in G,\ \text{and } ax = bx \text{ imply } \psi(a) = \psi(b).$$

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PROOF. The necessity of (*) is clear. For the sufficiency, we may suppose that G has a unit. By Zorn's lemma, it suffices to choose an $x_0 \in G-S$ and extend ψ to a character ψ_0 on $S_0 = \{sx_0^k : s \in S,\ k \geqq 0\}$ that satisfies (*) on S_0. Three cases must be considered. We omit the details which are similar to those in [**3**] but we note that in each case the crucial matters to be checked are that ψ_0 is well-defined and that ψ_0 satisfies (*) on S_0. In Cases 2 and 3, the denials of the previous cases are essential.

CASE 1. Suppose there exist $a_0,\ b_0 \in S$, $x_0 \in G-S$, and $y_0 \in G$ such that $a_0x_0y_0 = b_0y_0$. Then extend ψ to ψ_0 such that $\psi_0(x_0) = \psi(b_0)/\psi(a_0)$.

CASE 2. Suppose Case 1 does not apply but that for some $x_0 \in G-S$ and some $k_1 \geqq 2$, we have $x_0^{k_1} \in S$. Then let k_0 be the least positive integer such that $x_0^{k_0} \in S$ and extend ψ to ψ_0 such that $\psi_0(x_0)$ is any k_0th root of $\psi(x_0^{k_0})$.

CASE 3. Suppose Cases 1 and 2 do not apply. Then choose $x_0 \in G-S$ arbitrarily and extend ψ to ψ_0 so that $\psi_0(x_0) = 1$.

We now state the corollary implying [**1**, Theorem 4.2].

COROLLARY. *Let χ be a semicharacter on a commutative semigroup such that $\chi(x) = 0$ or $|\chi(x)| = 1$ for all $x \in G$. Let $S(\chi) = \{x \in G: |\chi(x)| = 1\}$ and suppose that A is a subsemigroup of G such that*

(1) $S(\chi) \subseteq A$;

(2) $x \in G,\ y \in G-A$ *imply* $xy \in G-A$;

(3) $x,\ y \in S(\chi),\ z \in A$, *and* $xz = yz$ *imply* $\chi(x) = \chi(y)$.

Then there is a semicharacter ψ on G such that $\{x \in G: |\psi(x)| = 1\} = A$ and $\chi(x) = \psi(x)$ for $x \in S(\chi)$.

PROOF. Let χ_0 be χ restricted to $S(\chi)$ and extend to a character ψ_0 on A using the preceding theorem. Then define $\psi(x) = \psi_0(x)$ for $x \in A$ and $\psi(x) = 0$ for $x \in G-A$.

NOTE. The above theorem and the theorem of [**3**] lead one to ask what conditions are necessary to extend semicharacters that never take the value zero. A natural conjecture would be condition (*) above and condition (A) of [**3**]:

$$\text{(A)} \qquad a,\ b \in S,\ x \in G, \text{ and } ax = b \text{ imply } |\psi(a)| \geqq |\psi(b)|.$$

However, consider the following example.[2] Let G be the commutative semigroup generated by $\{a_1, a_2, \cdots, b_1, b_2, \cdots, c, d\}$ and satisfying the relations:

$$(**) \qquad a_1^{k_1}a_2^{k_2} \cdots b_1^{m_1}b_2^{m_2} \cdots c^p d^q = a_1^{k_1'}a_2^{k_2'} \cdots b_1^{m_1'}b_2^{m_2'} \cdots c^{p'}d^{q'},$$

[2] Dr. Comfort suggested this simplification of the author's original example.

whenever $q>0$, $q'>0$, $k_n-m_n=k_n'-m_n'$ for all n, $q-p=q'-p'$, and $q+\sum_{n=1}^{\infty} m_n=q'+\sum_{n=1}^{\infty} m_n'$. (In the expression (**) all but finitely many of the exponents are zero.) Let S be the subsemigroup of G generated by $\{a_1, a_2, \cdots, c\}$ and define ψ on S by

$$\psi(a_1^{k_1}a_2^{k_2}\cdots c^p) = \prod_{n=1}^{\infty}\left(\frac{1}{n}\right)^{k_n}.$$

Then

(i) ψ never takes the value zero on S;

(ii) $a, b\in S$, $x\in G$, and $ax=bx$ imply $\psi(a)=\psi(b)$;

(iii) $a, b\in S$, $x, y\in G$, and $axy=by$ imply $|\psi(a)| \geqq |\psi(b)|$;

(iv) any extension of ψ to a semicharacter on G takes on the value zero.

Indeed, if ψ_0 extends ψ, then $\psi_0(d)=0$ since $a_nb_nd=cd^2$ implies that $|\psi_0(d)| \leqq 1/n$ for all n. The example can be considerably simplified if only condition (A) is desired rather than condition (iii).

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