ON F. SUPNICK'S SIX-CONIC THEOREM

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In Proc. Amer. Math. Soc. vol. 11 (1960) p. 498, F. Supnick proved that if P_1, P_2, P_3, P_4, P_5, P_6 are any six real coplanar points, not all on the same conic and no three collinear, then it is possible for each to be inside the conic through the other five, but impossible for each point to be outside. The complete theorem is that two, three, or six of the points lie inside the conic through the other five.

We call P_1 an "in-point" when it lies inside "conic I" (i.e., the conic $P_2P_3P_4P_5P_6$). We call it an "out-point" when it lies outside so that the tangents from P_1 to conic I are real; and so for the points P_2, etc. Since the theorem is unaltered by any real projection, there is no loss of generality in supposing that conic I is an ellipse with the points P_2, P_3, P_4, P_5, P_6 in order round its perimeter. Further projections can convert the ellipse into a circle with $P_3P_4P_5P_6$ a rectangle; which we may assume to be the case in what follows.

Suppose now that P_2, P_3, P_4, P_5, P_6 are kept fixed and that P_1 traces out a continuous path in the plane. When P_1 is very close to the line P_2P_3 (but not in the immediate neighbourhood of P_2 or P_3), conic VI approximates to the line-pair P_2P_3, P_4P_5 and it will be immediately evident from the diagram whether P_6 is an in-point or an out-point. Also it will be clear that as P_1 crosses the line P_2P_3 each of P_4, P_5, P_6 changes from in-point to out-point or vice versa, while P_1, P_2 and P_3 do not change in this manner. Similar results hold good when P_1 crosses P_2P_4, P_2P_5, etc. Again it will be seen that when P_1 crosses the perimeter of conic I each of the six points changes from in-point to out-point or vice versa. Using these facts we can readily determine whether each of the six points is an in-point or an out-point when P_1 lies in any one of those portions of the plane into which it is divided by the conic I and the ten lines which join two of the five points P_2, P_3, P_4, P_5, P_6; and thus verify the theorem.

For instance, each of the six points is an in-point if and only if P_1 lies in the pentagon bounded by P_2P_4, P_3P_5, P_4P_6, P_5P_2, P_6P_3.

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