SEMI-PEANIAN CHARACTERIZATIONS OF E^2 AND S^2

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The problem of characterizing the closed 2-cell E^2 as an irreducible lc-connexe about a 1-sphere using unicoherence and not assuming compactness was proposed by Professor R. L. Wilder. It is found in Theorem 1 that a semi-peanian space in which all 1-spheres are contained in lc-connexes which are irreducible relative to certain unicoherence properties and satisfy a certain incidence relation and which is itself such a connexe is indeed a closed 2-cell. In Theorem 2 there is obtained a characterization of the closed 2-sphere S^2 among the semi-peanian spaces.

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1. A space X will be called semi-compact [6, p. 327] if for every x in X and for every neighborhood $U(x)$ there is a neighborhood $V(x)$ which is contained in $U(x)$ and has a compact boundary. A semi-peanian space is connected, locally connected, semi-compact, complete metric and perfectly separable. A space X will be called unicoherent if it is connected and, for every pair of closed connected subsets Y, W of X, if X is equal to the union of Y and W, then the intersection of Y and W is connected. For further definitions see Wilder [5].

THEOREM 1. *Let R be a topological space satisfying the following conditions:*

(1) *To every* 1-*sphere T in R there corresponds a set $D(T)$ such that*

(1a) *T is contained in $D(T)$;*

(1b) *$D(T)$ is unicoherent and semi-peanian;*

(1c) *$D(T)$ is contained in every unicoherent and semi-peanian subset of R which contains T;*

(1d) *if x is a point of T, then $D(T)-x$ is unicoherent and semi-peanian:*

(1e) *if T and T' are* 1-*spheres in R whose intersection is an arc (ab), then one of the sets $D(T)$, $D(T')$ is contained in the other, or $D(T)\cdot D(T')=(ab)$ and the union of $D(T)$ with $D(T')$ is unicoherent and semi-peanian;*

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(2) *there is a* 1-*sphere* S *in* R *such that* $D(S)=R$.
Then R *is a closed* 2-*cell.*

LEMMA 1.1. *The space* R *has no cut points.*

If x is a point of S, then conditions (1d) and (2) imply that x is not a cut point of R. Suppose that x is a point of $R-S$ and that x is a cut point of R. Then $R-x=A+B$, separated, where A designates the component of S in $R-x$. Since R is a semi-peanian space, $A+x$ is also semi-peanian. Condition (1c) implies that $A+x$ is not unicoherent. Therefore $A+x=M+N$, where M, N are closed in $A+x$ and connected, and $M\cdot N=H+K$, separated. Suppose x is in M. Let $M'=M+B$ and $N'=N$. Then M' and N' are closed in R and connected, and $M'+N'=A+x+B=R$. Moreover, $M'\cdot N'=(M+B)\cdot N=H+K$, separated, since $B\cdot N=\varnothing$. Therefore R is not unicoherent, contrary to conditions (1b) and (2), and so the supposition that x is a cut point of R is false.

LEMMA 1.2. *If* A *is a closed subset of* R *which does not separate* R, *and* U *is an open connected subset of* R *containing* A, *then* $U-A$ *is connected.*

In view of Lemma 1.1, this lemma implies that no open connected subset of R has a cut point. Lemma 1.2 is easily proved by means of the Phragmén-Brouwer property [**5**, p. 47]. Neither A nor $R-U$ separates R, and so $R-A-(R-U)$ is connected, but of course this set is $U-A$.

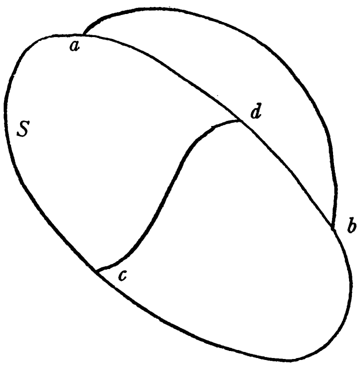
LEMMA 1.3. *If* T *and* T' *are distinct* 1-*spheres in* R, *then* $D(T)$ *and* $D(T')$ *are not equal.*

Since $T\neq T'$, there exists a point x in $T-T'$. If x is not in $D(T')$, then $D(T)\neq D(T')$. If x is in $D(T')$, then, by condition (1c), $D(T')-x$ is either not unicoherent or not semi-peanian. Lemma 1.1, applied to $D(T')$, implies that $D(T')-x$ is an open connected subset of $D(T')$. It follows that $D(T')-x$ is locally connected, semi-compact and perfectly separable. Moreover, $D(T)-x$ is a G_δ in a complete metric space and is therefore homeomorphic to a complete metric space. Therefore $D(T')-x$ is semi-peanian and consequently is not unicoherent. But x is in T, and, by condition (1d), $D(T)-x$ is unicoherent. Therefore $D(T)\neq D(T')$.

LEMMA 1.4. *There do not exist disjoint arcs* (ab) *and* (cd) *in* R *such that both arcs span* S, *and* a *and* b *separate* c *and* d *on* S.

To prove Lemma 1.4, suppose there exist such arcs. Let $\langle bca\rangle$ and

$\langle bda \rangle$ designate the components of c and d respectively in $S-(a+b)$. Also let $\langle dac \rangle$ and $\langle dbc \rangle$ be the components of a and b respectively in $S-(c+d)$. Let (ac) denote the arc from a to c on S not containing b or d. Let (ad) denote the arc from a to d on S not containing b or c. Let (bc) denote the arc from b to c on S not containing a or d, and let (bd) denote the arc from b to d on S not containing a or c. Then the following sets T_1 to T_6 are clearly 1-spheres:



$$T_1 = (ab) + (bca),$$
$$T_2 = (ab) + (bda),$$
$$T_3 = (cd) + (dbc),$$
$$T_4 = (cd) + (dac),$$
$$T_5 = (ab) + (bd) + (dc) + (ac),$$
$$T_6 = (ab) + (bc) + (cd) + (ad).$$

The plan of the proof is as follows: first show that $D(T_1)\cdot D(T_2) = (ab)$, and $D(T_1)+D(T_2)=R$; from this it is shown that $D(T_1)$ and $D(T_3)$ cannot satisfy condition (1e).

The intersection of T_1 with T_2 is (ab). Therefore, by condition (1e), one of the sets $D(T_1)$, $D(T_2)$ is contained in the other, or $D(T_1)\cdot D(T_2) = (ab)$ and $D(T_1)+D(T_2)$ is unicoherent and semi-peanian. If $D(T_1)$ is contained in $D(T_2)$, then S is contained in $D(T_2)$, and, by conditions (1c) and (2), so is R. Consequently $R=D(T_2)$. But Lemma 1.3 implies that R does not equal $D(T_2)$. Therefore $D(T_1)$ is not contained in $D(T_2)$, and, by similar reasoning, $D(T_2)$ is not contained in $D(T_1)$. Then $D(T_1)\cdot D(T_2)=(ab)$, and $D(T_1)+D(T_2)$ is unicoherent and semi-peanian. Since S is contained in this union, condition (1c) implies that R is also contained therein. Therefore $D(T_1)+D(T_2)=R$.

The intersection of T_1 with T_3 is (bc). Suppose $D(T_3)$ is contained in $D(T_1)$. The intersection of T_1 with T_6 is $(ab)+(bc)=(abc)$, but, since (cd) is contained in $D(T_1)$, $D(T_1)\cdot D(T_6) \neq (abc)$. Therefore one of the sets $D(T_1)$, $D(T_6)$ is contained in the other. If $D(T_1)$ is contained in $D(T_6)$, then, under our present assumption, $D(T_3)$ is also contained in $D(T_6)$, and $R=D(T_6)$, contrary to Lemma 1.3. If $D(T_6)$ is contained in $D(T_1)$, then, under our supposition, S is contained in $D(T_1)$, and $R=D(T_1)$, which is impossible. Therefore the supposition that $D(T_3)$ is contained in $D(T_1)$ is false.

Now suppose that $D(T_1)$ is contained in $D(T_3)$. Then one of the sets $D(T_3)$, $D(T_6)$ is contained in the other, and again Lemma 1.3 is violated. Therefore $D(T_1)$ is not contained in $D(T_3)$.

Then necessarily $D(T_1)\cdot D(T_3)=(bc)$. Since $D(T_1)\cdot D(T_2)=(ab)$, and R is equal to $D(T_1)+D(T_2)$, $D(T_3)-(bc)$ is contained in $D(T_2)$. Then $D(T_3)$ is contained in $D(T_2)$, or $D(T_2)$ is contained in $D(T_3)$. The first case implies that (bc) is contained in $D(T_2)$ contrary to the fact that $D(T_1)\cdot D(T_2)=(ab)$. The second case implies that (ab) is contained in $D(T_3)$ contrary to the fact that the intersection of $D(T_1)$ with $D(T_3)$ is equal to (bc). Therefore $D(T_1)$ and $D(T_3)$ do not satisfy condition (1e), and so there exist no such arcs as (ab) and (cd).

LEMMA 1.5. *If T is a 1-sphere in R, and (ab) is an arc contained in T, then $D(T)-(ab)$ is connected.*

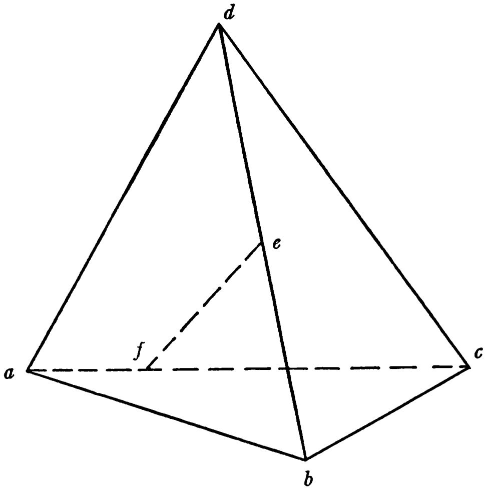
To prove Lemma 1.5, let M designate a component of $D(T)-(ab)$, and let $F(M)$ denote the boundary of M and $\overline{M}$ denote the closure of M in $D(T)$. If $D(T)-(ab)$ is not connected, let C be a component of $D(T)-\overline{M}$ such that $F(C)\cdot F(M)$ contains a nondegenerate arc $(c'd')$. Let p be a point in $\langle c'd'\rangle$. Then $\overline{M}-p$ is connected and closed in $D(T)-p$ as is the complement of M in $D(T)-p$. Moreover, $D(T)-p$ is equal to the union of $\overline{M}-p$ with $D(T)-p-M$, and $(\overline{M}-p)\cdot(D(T)-p-M)$ is contained in $(cp\rangle+\langle pd)$ and has a non-null intersection with both $(c'p\rangle$ and $\langle pd')$ and is therefore not connected. This implies that $D(T)-p$ is not unicoherent, contrary to condition (1d). Therefore $C=\varnothing$, and $D(T)-(ab)$ is connected.

LEMMA 1.6. *If (ab) is an arc spanning S, then $R-(ab)$ has exactly two components.*

As a result of Lemma 1.4, (ab) separates R. Let c, d be points on S such that a, b separate c, d on S so that $S=(bca)+(bda)$. Then $(ab)+(bca)$ is a 1-sphere T_1, and $(ab)+(bda)$ is a 1-sphere T_2. As in the proof of Lemma 1.4, $D(T_1)\cdot D(T_2)=(ab)$, and $D(T_1)+D(T_2)=R$. So $R-(ab)$ is equal to $(D(T_1)-(ab))+(D(T_2)-(ab))$, and, by Lemma 1.5, $D(T_1)-(ab)$ and $D(T_2)-(ab)$ are connected. Therefore $R-(ab)$ has exactly two components.

LEMMA 1.7. *The space R does not contain any primitive skew curve of type* I *or type* II [**3**, vol. II, p. 230].

A primitive skew curve consists of seven arcs (ab), (bc), (ca), (ad), (bd), (cd) and (ef) with intersections as indicated in the accompanying sketch. Let



$T_1 = (afeda)$,
$T_2 = (bcfeb)$,
$T_3 = (cfedc)$,
$T_4 = (afeba)$.

Condition (1e) of the hypothesis of Theorem 1 implies, concerning $D(T_1)$ and $D(T_4)$, that $D(T_1)\cdot D(T_4)=(afe)$ and $D(T_1)+D(T_4)$ is unicoherent and semi-peanian or $D(T_1)$ is contained in $D(T_4)$ or $D(T_4)$ is contained in $D(T_1)$. The same condition implies, concerning $D(T_2)$ and $D(T_3)$, that $D(T_2)\cdot D(T_3)=(cfe)$ and $D(T_2)+D(T_3)$ is unicoherent and semi-peanian or $D(T_2)$ is contained in $D(T_3)$ or $D(T_3)$ is contained in $D(T_2)$. The conjunction of the two 3-term disjunctions of the two preceding sentences can be written as a disjunction of nine terms. These terms are the following:

(1) $D(T_1)\cdot D(T_4)=(afe)$ and $D(T_1)+D(T_4)$ is unicoherent and semi-peanian and $D(T_2)\cdot D(T_3)=(cfe)$ and $D(T_2)+D(T_3)$ is unicoherent and semi-peanian;

(2) $D(T_1)\cdot D(T_4)=(afe)$ and $D(T_1)+D(T_4)$ is unicoherent and semi-peanian and $D(T_2)$ is contained in $D(T_3)$;

(3) $D(T_1)\cdot D(T_4)=(afe)$ and $D(T_1)+D(T_4)$ is unicoherent and semi-peanian and $D(T_3)$ is contained in $D(T_2)$;

(4) $D(T_1)$ is contained in $D(T_4)$ and $D(T_2)\cdot D(T_3)=(cfe)$ and $D(T_2)+D(T_3)$ is unicoherent and semi-peanian;

(5) $D(T_1)$ is contained in $D(T_4)$ and $D(T_2)$ is contained in $D(T_3)$;

(6) $D(T_1)$ is contained in $D(T_4)$ and $D(T_3)$ is contained in $D(T_2)$;

(7) $D(T_4)$ is contained in $D(T_1)$ and $D(T_2)\cdot D(T_3)=(cfe)$ and $D(T_2)+D(T_3)$ is unicoherent and semi-peanian;

(8) $D(T_4)$ is contained in $D(T_1)$ and $D(T_2)$ is contained in $D(T_3)$;

(9) $D(T_4)$ is contained in $D(T_1)$ and $D(T_3)$ is contained in $D(T_2)$.

It will be shown that each of these nine statements is false, and therefore there can be no primitive skew curve of type I in R.

Condition (1) implies $D(abda)=D(T_1)+D(T_4)$ and that $D(bcdb)=D(T_2)+D(T_3)$. Then (ef) is contained in $D(abda)\cdot D(bcdb)$ and therefore one of these two sets is contained in the other. But under either inclusion relation there are present arcs which violate Lemma 1.4, and so condition (1) is false.

One argument suffices to show that conditions (2), (3), (4) and (7) are false. The argument will be made relative to condition (2). Condition (2) implies that $D(abda)=D(T_1)+D(T_4)$ and $D(bcdb)$ is contained in $D(T_3)$. Since (ef) is contained in the intersection of $D(abda)$ with $D(T_3)$, it follows that one of these two sets is contained in the other. If $D(abda)$ is contained in $D(T_3)$, then (fad) and (cbe) are arcs of $D(T_3)$ which span T_3 and which violate Lemma 1.4. If $D(T_3)$ is contained in $D(abda)$, then (afe) and (bcd) are arcs of $D(abda)$ which violate Lemma 1.4. Therefore condition (2) is false, and, by precisely the same type of argument, so are conditions (3), (4) and (7).

Conditions (5) and (9) can be shown to be false by one argument. The argument will be made relative to condition (5). Condition (5) implies that $D(abda)$ is contained in $D(T_4)$ and $D(bcdb)$ is contained in $D(T_3)$. If, in addition, $D(T_4)$ is contained in $D(T_3)$, then (ad) and (ebc) are arcs of $D(T_3)$ which violate Lemma 1.4. For similar reasons, $D(T_3)$ is not contained in $D(T_4)$. Therefore the intersection of $D(T_3)$ with $D(T_4)$ is (ef). But b is in $D(T_2)$ and $D(T_2)$ is contained in $D(T_3)$. Since b is also in $D(T_4)$, $D(T_4)\cdot D(T_3)\neq(ef)$, contrary to condition (1e) of the hypothesis of Theorem 1. Therefore conditon (5) is false, and, by a similar argument, so is condition (9).

One argument suffices to show that the remaining conditions, (6) and (8), are also false. Condition (6) implies that $D(abda)$ is contained in $D(T_4)$ and $D(bcdb)$ is contained in $D(T_2)$. If $D(T_4)$ is contained in $D(T_2)$, then (fab) and (edc) are arcs of $D(T_2)$ which violate Lemma 1.4. If $D(T_2)$ is contained in $D(T_4)$, then (bcf) and (eda) are arcs of $D(T_4)$ which violate Lemma 1.4. Then necessarily $D(T_4)\cdot D(T_2)=(bef)$. But condition (6) implies that d is in $D(T_4)\cdot D(T_2)$. Therefore condition (6) is false, and, by a similar argument, so is condition (8).

By an entirely analogous argument it may be shown that there is no primitive skew curve of type II in R.

This completes the proof of Lemma 1.7. As a consequence of Lemma 1.7, R can be imbedded in S^2 [6, pp. 339–340]. By consideration of an arc spanning S and by means of Lemmas 1.4 and 1.5, it is easy to show that $R-S$ is connected. The image of S in S^2 is a 1-sphere J which separates S^2 into two components with $R-S$ in one of them.

This component A and its boundary can be mapped homeomorphically by stereographic projection onto a subset C of the plane E consisting of a 1-sphere and the bounded component of its complement in E. There exists [4, Theorem 4.1, p. 165] a homeomorphism from C onto a closed square region G in E, and therefore a homeomorphism f from R into G such that the image of S is the boundary of G. The region G is the set of points (x, y) such that $|x| \leqq 1$ and $|y| \leqq 1$.

If there exists a point q with coordinates (x', y') in $G-f(R)$, then let L be the straight line consisting of the points (x, y) such that $|x| \leqq 1$ and $y=y'$. Let N be the component of $(0, -1)$ in $f(R)-L$, and let M be the component of $(0, 1)$ in the complement of the closure of N in $f(R)$. Then $F(M)$ is contained in L and contains $(-1, y')$ and $(1, y')$ but not (x', y') and is therefore not connected. This implies that R is not unicoherent, and so $G-f(R)$ is empty.

2. THEOREM 2. *Let R be a unicoherent semi-peanian space such that for every subset A in R which is either a point or an arc, $R-A$ is unicoherent, and for every pair of points x, y in R, $R-y-x$ is not unicoherent. Then R is homeomorphic to the sphere S^2.*

LEMMA 2.1. *For every pair of points y, z in R there exists a 1-sphere $J(y, z)$ which separates y and z in R.*

By hypothesis, $R-y-z$ is not unicoherent, and so there exists [2, p. 184] a 1-sphere $J(y, z)$ which is a retract of $R-y-z$. If there exists an arc (yz) in $R-J(y, z)$, then the same function which retracts $R-y-z$ onto $J(y, z)$ also retracts $R-(yz)$ onto $J(y, z)$. But, by hypothesis, $R-(yz)$ is unicoherent and so cannot be retracted onto $J(y, z)$. Therefore there exists no such arc as (yz). Since every component of the complement of $J(y, z)$ is arcwise connected, y and z must be in different components of $R-J(y, z)$.

For Lemmas 2.2, 2.3, and 2.4, let J be a 1-sphere which separates R, and let (ab) be an arc spanning J so that $J=(acb)+(bda)$. Let K designate the 1-sphere $(acb)+(ab)$, and let $L=(bda)+(ab)$.

LEMMA 2.2. *Either K or L separates R.*

$R-J=A+B$, separated, and, since R is unicoherent, $\langle ab\rangle$ does not contain any component of $R-J$. Therefore $R-J-(ab)=M+N$, separated, and clearly $R-J-(ab)=R-K-L$. The open arcs $\langle acb\rangle$ and $\langle bda\rangle$ are disjoint closed subsets of $R-(ab)$. Since $R-(ab)$ is unicoherent, and $\langle acb\rangle+\langle bda\rangle$ separates $R-(ab)$, then necessarily either $\langle acb\rangle$ or $\langle bda\rangle$ separates $R-(ab)$, and so K or L separates R.

LEMMA 2.3. *The open arc $\langle ab\rangle$ separates the component of $R-J$ in which it lies.*

Let C designate the component of $\langle ab\rangle$ in $R-J$. The hypothesis of Theorem 2 easily implies that, for every component D of $R-J$, $F(D)=J$. If $C-\langle ab\rangle$ is connected, then $(C-\langle ab\rangle)+\langle bda\rangle$ is connected, and so is $\langle bda\rangle+(R-\overline{C})$. Then, since $R-K=(C-\langle ab\rangle)+\langle bda\rangle+(R-\overline{C})$, $R-K$ is connected, and, by similar reasoning, so is $R-L$. But this contradicts Lemma 2.2, and so $\langle ab\rangle$ separates C.

LEMMA 2.4. *Both K and L separate R.*

Suppose K separates R. By Lemma 2.3, $C-(ab)$ is not connected. Let D be a component of $C-(ab)$. Then D is also a component of $R-J-(ab)$, and $F(D)$ is a closed connected subset of $J+(ab)$. Since $F(C)=J$, some component E of $C-(ab)$ has limit points in $\langle bda\rangle$. Since E also has limit points in $\langle ab\rangle$, $F(E)$ contains L. If $F(E)\cdot\langle acb\rangle=\varnothing$, then L separates R. If $F(E)\cdot\langle acb\rangle\neq\varnothing$, then $E+\langle acb\rangle+\langle bda\rangle$ is a closed connected subset of $R-(ab)$. Let $H=E+\langle acb\rangle+\langle bda\rangle$, and let B be any component of $R-J$ other than C. Then B is also a component of $R-(ab)-H$. But since the frontier of B in R is equal to J, the frontier of B in $R-(ab)$ is equal to $\langle acb\rangle+\langle bda\rangle$, and is therefore not connected. But, by the hypothesis of Theorem 2, $R-(ab)$ is unicoherent. Therefore $F(E)\cdot\langle acb\rangle=\varnothing$, and L separates R.

LEMMA 2.5. *For every point y in R there exists a neighborhood $N(y)$ such that for every* 1-*sphere J contained in $N(y)$, J separates R.*

By Lemma 2.1, for every pair of points y, z in R there exists a 1-sphere $K(y, z)$ which separates y and z in R. Let $N(y)$ be a neighborhood of y such that $N(y)\cdot K=\varnothing$. Let J be any 1-sphere contained in $N(y)$. Since R is semi-peanian and not separated by any arc, there exist disjoint arcs (ab) and (cd) such that $J\cdot((ab)+(cd))=a+c$, and $((ab)+(cd))\cdot K=b+d$. Let points e and f be in different components of $J-a-c$ so that $J=(aec)+(afc)$. Let g and h be in different components of $K-b-d$ so that $K=(bgd)+(bhd)$. Let L be the union of the arcs (afc), (ab), (bgd) and (cd). The 1-sphere K is spanned by $(afc)+(ab)+(cd)$, and (bgd) is contained in K, and so, by Lemma 2.4, L separates R. The arc (aec) spans L, and (afc) is contained in L, and so, again by Lemma 2.4, $(aec)+(afc)$ separates R. Therefore J separates R.

LEMMA 2.6. *Every* 1-*sphere in R separates R.*

Let J be a 1-sphere in R, and let y, z be points in $R-J$. Let $N(y)$ be a neighborhood of y such that $N(y)\cdot J=\varnothing$, and every 1-sphere

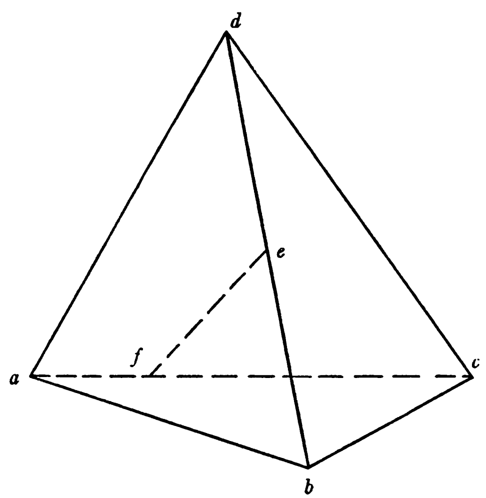
H in $N(y)$ separates R. Let H be a 1-sphere in $N(y)$. Then, by repeating the argument of Lemma 2.5, it is seen that J separates R.

LEMMA 2.7. *If K is an open or half open arc in R, and K is closed in R, then $R-K$ is connected.*

Suppose $R-K$ is not connected, and let C be a component of $R-K$. Since R is unicoherent and semi-peanian, the boundary of C is a point, an arc, a half open arc, or all of K. By the hypothesis of Theorem 2, the boundary cannot be a point or an arc. In either of the two remaining cases there must exist a component D of $R-K$ and an arc (ab) such that (ab) is contained in $F(C)\cdot F(D)$. Let p be a point of the open arc $(ab)-a-b$. By hypothesis, $R-p$ is unicoherent. However, $\overline{D}-p$ is a closed connected subset of $R-p$, and C is a component of $R-p-(\overline{D}-p)$. But the frontier of C in $R-p$ is contained in K and contains both $(ap\rangle$ and $\langle pb)$ and is therefore not connected. It follows that $R-K$ is connected.

LEMMA 2.8. *There are no primitive skew curves of type* I *or* II *in R.*

Suppose there were a primitive skew curve C of type I in R. By Lemma 2.6, the 1-sphere $(abcda)$ separates R, and, by Lemma 2.3, the open arc $\langle ac\rangle$ separates the component A of $R-(abcda)$ in which it lies, and no component of $A-\langle ac\rangle$ has limit points in both $\langle adc\rangle$ and $\langle abc\rangle$. But $\langle fe)+\langle bd\rangle$ is contained in $A-\langle ac\rangle$ and has limit points b and d. Therefore there is no such curve as C in R.



In an entirely similar manner it can be shown that there is no primitive skew curve of type II in R.

By [6, pp. 339–340], R can be imbedded in S^2. Let f be a homeomorphism from R to S^2. If $S^2-f(R)\neq\emptyset$, then let K be a 1-sphere of S^2 which separates $f(R)$ and contains at least one point of $S^2-f(R)$. S^2-K consists of two domains, A and B. Let C be a component of $A\cdot f(R)$, and let $\overline{C}$ denote the closure of C in $f(R)$. $\overline{C}$ is a closed connected subset of $f(R)$. Let D be a component of $f(R)-\overline{C}$. Since $f(R)$ is locally connected and unicoherent, the frontier of D in $f(R)$ is a closed connected subset of $\overline{C}$. The frontier of D is contained in K. But K is not contained in $f(R)$, and, by Lemma 2.7, no proper connected subset of K separates $f(R)$. Therefore $S^2-f(R)=\emptyset$. This completes the proof of Theorem 2.

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