LIMITS OF SEQUENCES OF FINITELY GENERATED ABELIAN GROUPS

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1. **Introduction.** The limits referred to in the title are the ordinary direct and inverse limits, but for our purpose we shall require throughout §§2 and 3 that the homomorphisms associated with direct limits be monomorphisms and that those associated with inverse limits be epimorphisms. Denote by $\mathcal{C}_D$ ($\mathcal{C}_I$) the collection of sequences of groups for which a direct (inverse) limit exists and is determined up to isomorphism solely by the groups, that is, is independent of the choice of the monomorphisms (epimorphisms). In §§2 and 3, respectively, the sequences of finitely generated abelian groups belonging to $\mathcal{C}_D$ and $\mathcal{C}_I$ are determined. A characterization of inverse limits of sequences of finitely generated abelian groups is given in §4.

Since our attention will be restricted to abelian groups, "abelian group" is abbreviated to "group". With one exception, all groups are considered discrete.

2. **Direct limits.** We wish to consider only sequences G_n which admit monomorphisms from G_n to G_{n+1}; such sequences will be called ascending. A basis $[g_1, g_2, \cdots, g_k]$ of a finite primary group is said to be written in canonical form if its elements are arranged in decreasing order, $O(g_{i+1}) \leqq O(g_i)$ for $1 \leqq i < k$ where $O(x)$ denotes the order of x. Let G_n be an ascending sequence of finite p-primary groups with canonical bases $[g_{1,n}, g_{2,n}, \cdots, g_{k(n),n}]$. Notice that the sequences $k(n)$ and $O(g_{i,n})$ (with i fixed) are monotonic since G_n is ascending. The rank (finite or ∞) of the sequence G_n is defined as $r = \lim_{n\to\infty} k(n)$, and for each natural integer $i \leqq r$, m_i is defined by $p^{m_i} = \lim_{n\to\infty} O(g_{i,n})$. Set $m = \min \{m_i\}$.

The sequence G_n is called irregular if for some fixed $m_0 < m$ the relation $O(g_{i,n}) = p^{m_0}$ holds for almost all n with a suitable choice of i, that is, G_n is irregular if all but a finite number of the groups G_n contain a summand of the same order $p^{m_0} < p^m$. Sequences are said to be regular if they are not irregular. Obviously, if the rank of the sequence G_n is finite, then G_n is regular.

THEOREM 1. *A necessary and sufficient condition for an ascending sequence G_n of finite primary groups to belong to $\mathcal{C}_D$ is that G_n be regular.*

PROOF. The fact that the condition is necessary is almost trivial.

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The most natural[2] direct limit of G_n is $A = \sum \oplus C(p^{m_i})$, where $C(p^{m_i})$ is the cyclic group of order p^{m_i} for finite m_i and $C(p^\infty)$ is quasicyclic (=group of type p^∞). However, if G_n is irregular, almost all the groups G_n contain a cyclic summand (isomorphic to) $C(p^{m_0})$ where $m_0 < m$; clearly, this summand can be preserved (as a summand) in some direct limit. Since such a limit cannot be isomorphic to A, $G_n \notin \mathcal{C}_D$.

In order to prove that regularity is sufficient, let G_n be regular and let B be an arbitrary direct limit of G_n. Use is made of the limit A defined above; we show that $B \cong A$. Recall that a direct limit L of G_n (under monomorphisms) is the set-theoretical union of groups isomorphic to the G_n; $L = \bigcup H_n$ where $H_n \cong G_n$ and $H_n \subseteq H_{n+1}$. Thus it is immediate (whether G_n is regular or not) that the rank (for definition, see [1]) of B is precisely the rank of the sequence G_n, $r(B) = r$.

If $m = \infty$, then B is divisible; for otherwise B would have a cyclic summand C ($\neq 0$), which would imply that almost all the G_n contained a summand isomorphic to C. Since G_n is regular, this is impossible. Thus $B \cong A$ since both are divisible with the same rank, and the theorem is proved in the case $m = \infty$.

For the case $m < \infty$, let $B = B_R \oplus B_D$ where B_R is reduced and B_D is divisible. B_R is bounded since B_R unbounded implies that $B_R = C_1 \oplus C_2 \oplus \cdots \oplus C_k \oplus B_R'$, where k and the orders of the cyclic groups C_i are arbitrarily large; but this implies $m = \infty$. Hence B_R is a direct sum of cyclic groups, and $B = B_1 \oplus B_2 \oplus \cdots \oplus B_D$ where B_i is a direct sum of cyclic groups of order p^i. Since G_n is regular, $B_1 = B_2 = \cdots = B_{m-1} = 0$. Using the corresponding notation, we write $A = A_m \oplus A_{m+1} \oplus \cdots \oplus A_D$.

It is immediate that $p^t B$ ($t \geqq 0$) is a direct limit of the sequence $p^t G_n$; hence the rank of $p^t B$ is determined solely by the groups G_n. Thus $A_D \cong B_D$ since the reduced parts of A and B are bounded. Moreover, $r(p^m B) = r(p^m A) < \infty$, and

$$r(A_m) = r(A) - r(p^m A) = r(B) - r(p^m B) = r(B_m);$$

hence $A_m \cong B_m$. Similarly,

$$r(A_{m+1}) = r(p^m B) - r(p^{m+1} B) = r(B_{m+1});$$

$A_{m+1} \cong B_{m+1}$, etc.

Since any subgroup $\neq 0$ of the additive rationals is a direct limit of infinite cyclic groups, we have at once

COROLLARY 1. *An ascending sequence G_n of finitely generated groups belongs to $\mathcal{C}_D$ if and only if the groups G_n are finite and the p-primary components form a regular sequence for each prime p.*

[2] The limit A is obtained merely by identifying $g_{i,n}$ with a multiple of $g_{i,n+1}$.

3. **Inverse limits.** Here, we wish to consider only sequences G_n which admit epimorphisms from G_{n+1} to G_n; such sequences will be called extending. However, a sequence of finite (abelian) groups is extending if and only if it is ascending; hence the terminology established in §2 carries over to extending sequences of finite p-groups.

THEOREM 2. *A necessary and sufficient condition for an extending sequence G_n of finite primary groups to belong to $\mathcal{C}_I$ is that G_n be regular.*

PROOF. Again, it is immediate that the condition is necessary. In fact, one needs only to make the obvious replacements in the proof of Theorem 1. Recall that $\sum\oplus$, the direct sum, should be replaced by $\sum'\oplus$, the complete direct sum, and the group of type p^∞ by the p-adic group.

Denote by G^* the (discrete) character group of G. With an inverse sequence

$$G_1 \leftarrow G_2 \leftarrow \cdots \leftarrow G_n \leftarrow \cdots \tag{1}$$

there is associated in a natural way a (unique) direct sequence

$$G_1^* \rightarrow G_2^* \rightarrow \cdots \rightarrow G_n^* \rightarrow \cdots, \tag{2}$$

which in turn gives rise to an inverse sequence

$$G_1^{**} \leftarrow G_2^{**} \leftarrow \cdots \leftarrow G_n^{**} \leftarrow \cdots. \tag{3}$$

Together with the usual isomorphism (see, for example, [2]) between G_n and G_n^{**}, (1) and (3) form a commutative diagram and therefore have isomorphic limits. However, the limit of (3) is the character group of the limit of (2). Thus if G_n is regular, by Theorem 1 the limit of (1) is determined by the groups alone and $G_n \in \mathcal{C}_I$.

Now let G_n be an extending sequence of finitely generated groups. Analogous to the p-rank of a sequence, the torsion free rank of the sequence G_n is defined as the limit of the torsion free ranks of the groups G_n, $r_0 = \lim_{n\to\infty} r_0(G_n)$. The question of whether or not G_n is contained in $\mathcal{C}_I$ depends to a large degree on whether r_0 is finite or not; consequently, the two cases are distinguished.

COROLLARY 2. *Let G_n be an extending sequence with finite torsion free rank of finitely generated groups. A necessary and sufficient condition for G_n to belong to $\mathcal{C}_I$ is that the p-primary components form a regular sequence for each prime p.*

PROOF. Denoting the torsion free rank of the sequence G_n by r_0, we may as well assume that the torsion free rank of the group G_n is r_0 for each n. Clearly, an inverse limit of G_n is the free group of rank

r_0 plus a limit of the torsion subgroups of G_n. The proof is completed by Theorem 2.

COROLLARY 3. *Let G_n be an extending sequence with infinite torsion free rank of finitely generated groups. A necessary and sufficient condition for G_n to belong to $\mathcal{C}_I$ is that the torsion subgroups of G_n do not contribute to any inverse limit of G_n, that is, there is no nonzero limit element $\{x_n\}$ where x_n is torsion for all n.*

PROOF. It is clear that the complete direct sum of a countably infinite number of infinite cyclic groups is an inverse limit of G_n; denote this group by A. Let B be an arbitrary inverse limit of G_n and let C denote the subgroup of B which comes from the torsion subgroups, T_n, of G_n. Then B/C, isomorphic to an inverse limit of G_n/T_n (since the T_n are finite), is isomorphic to A. Hence if $C=0$, then $B\cong A$; but if $C\neq 0$, then it contains either a cyclic group of prime order or a p-adic group, so B is not isomorphic to A.

4. **Totality of limits.** The requirement that the homomorphisms associated with direct and inverse limits be monomorphisms and epimorphisms, respectively, is dropped. It is immediate that the direct limits of sequences of finitely generated groups are just the countable groups. The class of inverse limits of such sequences is given by the following

THEOREM 3. *A necessary and sufficient condition for a group G to be an inverse limit of a sequence of finitely generated (abelian) groups is that G be the complete direct sum of (at most) a countable number of cyclic and p-adic groups.*

PROOF. It is obvious that the condition is sufficient. In order to prove that it is necessary, let G be an inverse limit of a sequence G_n of finitely generated groups and let C denote the subgroup of G which comes from the torsion subgroups, T_n, of G_n. The factor group G/C, isomorphic to an inverse limit of G_n/T_n, is the complete direct sum of a countable number of infinite cyclic groups. It is well known that C admits a compact topology in which it is totally disconnected and satisfies the second axiom of countability. Since the Pontrjagin dual group C^* of C is countable, it follows from the proof of Proposition 3.1 of [**3**] that C is a complete direct sum of a countable number of cyclic and p-adic groups. Since C is compact, it is algebraically compact [**4**]; hence its purity implies that it is a direct summand of G, and the theorem is proved.

5. **Remark and example.** In conclusion we add the following remark relevant to §§2 and 3. There are direct (inverse) sequences,

even of finite groups, which have two nonisomorphic limits associated with monomorphisms (epimorphisms) ρ_n and ϕ_n, respectively, such that $\rho_n \neq \phi_n$ only if both ρ_n and ϕ_n are onto isomorphisms. In fact, let G be a countable, reduced p-primary group with (nonzero) elements of infinite height; an increasing sequence of finite subgroups G_n which leads up to G exhibits such a sequence if the G_n are chosen such that the index of G_n in G_{n+1} is 0 or p, depending on whether n is odd or even.

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