

A POISSON KERNEL FOR CERTAIN HOMOGENEOUS SPACES

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1. Introduction. The aim of this note is to prove some of the results of D. Lowdenslager [6] on potential theory in symmetric, bounded domains in a more general setting. In particular, our methods are not restricted to complex manifolds nor to spaces imbeddable in Euclidean space.

The main result is Theorem 3.1, which presents a Lie group-theoretic method for constructing a Poisson kernel for certain homogeneous spaces G/K , with G a noncompact Lie group, K a compact, connected subgroup.

In section 2 we present several remarks that are meant to indicate how some of the results in [6] on the Bergman-Šilov boundary of a Cartan domain can be adapted to other situations, namely when a homogeneous space is equivariantly imbedded in a bigger space. Unfortunately, we do not yet know of any examples other than the Cartan domains where all of this machinery can be applied, although there are indications of this possibility in Karpelevič's work [5]. However, it is at least evident that results such as these that are not so dependent on explicit calculation as Lowdenslager's [6] are useful in dealing with the exceptional Cartan domains.

2. The Bergman-Šilov boundary. All manifolds, Lie groups, action of Lie groups on manifolds, tensor-fields, maps, etc. will be of differentiability class C^∞ unless mentioned otherwise. We follow Chevalley [3] for Lie group and differential-geometric notations, with some of the modifications suggested in [1]. All manifolds will be assumed to be connected and paracompact. Let M be such a manifold. For $x \in M$, M_x is the tangent space to M at x . If $\phi: M \rightarrow M'$ is a map of manifolds, $\phi_*: M_x \rightarrow M'_{\phi(x)}$ denotes the linear map that ϕ indicates on tangent vectors. If ω is a differential form on M' , $\phi^*(\omega')$ denotes the form on M induced by the map ϕ .

Let G be a Lie group that acts on M . This means that, for each $g \in G$, there is a diffeomorphism $T_g: M \rightarrow M$ such that:

- (a) $T_{g_1 g_2} = T_{g_1} T_{g_2}$ for $g_1, g_2 \in G$.
- (b) $T_e = \text{identity map of } M$, where e is the identity element of G .

We write:

$T_g(x) = gx$ or $g \cdot x$ for $g \in G$, $x \in M$. Let \mathfrak{G} be the Lie algebra of G .

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Let $V(M)$ be the set of vector fields on M , considered as an infinite dimensional Lie algebra with respect to the Jacobi bracket operation.

If $X \in V(M)$, and ω is a differential form on M , let $X(\omega)$ denote the Lie derivative of ω by the vector field X . (This is denoted by $\theta(X)(\omega)$ in [2], to which we also refer for the definition of this operation.)

Let $C^\infty(M)$ be the ring C^∞ real-valued functions on M . The operation of Lie derivation with respect to the vector field X on M defines a derivation of $C^\infty(M)$, i.e., X defines a first-order, linear differential operator on M .

More or less as definition of the Jacobi-bracket operation, we have:

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \text{ for } X, Y \in V(M), f \in C^\infty(M).$$

The action of G on M defines a Lie-algebra homomorphism $\rho: \mathbf{G} \rightarrow V(M)$. As definition:

For $X \in \mathbf{G}$, $x \in M$, $\rho(X)(x)$ is the tangent vector to the curve $t \mapsto \text{Exp}(tX)x$ at $t=0$. One shows that:

$$(2.1) \quad \text{For } g \in G, X \in \mathbf{G}, f \in C^\infty(M), T_g^*(\rho(X)(f)) = \rho(\text{Ad } g(X))T_g^*(f).$$

For $x \in M$, let Gx be the orbit of G at x , L_x be the isotropy group of G at x . Gx is isomorphic in a natural way to G/L_x , the space of right cosets.

If S is a subset of M such that $G \cdot S \subset S$, define $C^\infty(S)$ (resp. $\mathbf{C}^\infty(S)$) as the ring of all real valued, continuous (resp. not necessarily continuous) functions that are C^∞ when restricted to the orbits of G on S .

We suppose that \mathbf{G} has a nondegenerate quadratic form (\cdot, \cdot) that is invariant under $\text{Ad } G$. Choose a basis for \mathbf{G} ,

$$(X_i), 1 \leq i \leq n, \text{ such that}$$

$$(2.2) \quad (X_i, X_j) = 0 \text{ if } i \neq j.$$

Set $A_i = (X_i, X_i)$. Define the second-order linear differential operation Δ on $C^\infty(M)$ as follows:

$$(2.3) \quad \Delta(f) = \sum_{i=1}^n A_i \rho(X_i) \rho(X_i)(f), \quad \text{for } f \in C^\infty(M).$$

It is clear that it is independent of the basis chosen for \mathbf{G} satisfying 2.2. Also, note that 2.1 implies that Δ is invariant under the action of G , i.e.,

$$T_g^*(\Delta(f)) = \Delta T_g^*(f) \quad \text{for } f \in C^\infty(M), G \in G.$$

Note that:

$$\text{For } X \in \mathbf{G}, f \in C^\infty(M), \rho(X)(\rho(X)(f))(x) = \left. \frac{\partial^2}{\partial t^2} f(\text{Exp}(Xt) \cdot x) \right|_{t=0}.$$

In particular, if $X \in L_x$, the Lie algebra of the isotropy group at x , then $\rho(X)(\rho(X)(f))(x) = 0$.

Suppose that the basis (X_i) is chosen so that X_i , $1 \leq i \leq m$ spans L_x , and X_i , $m+1 \leq i \leq n$ spans M_x , the orthogonal complement of L_x with respect to $(\ , \)$. Then,

$$(2.4) \quad \Delta(f)(x) = \sum_{i=m+1}^n A_i \rho(X_i)(\rho(X_i)(f))(x).$$

As an application of these remarks, suppose that the form $(\ , \)$ restricted to each M_x is positive definite and G is transitive on M , i.e., the map $X \rightarrow \rho(X)(x)$ of $G \rightarrow M_x$ is onto with kernel L_x . This defines an isomorphism of M_x with M_x , hence a positive-definite quadratic form on M_x , i.e., M is a Riemannian manifold, with G acting as a group of isometries. It is known [7] that, for each $X \in M_x$, the curve $t \rightarrow \text{Exp}(tX) \cdot x$ is a geodesic. Then 2.4, together with the classical formula for the Laplacian operator in Riemann normal coordinates, prove the following result:

PROPOSITION 2.1. *With the assumptions described above, Δ is the Laplace operator of the Riemannian metric on M invariant under G .*

We return now to the case where G is not necessarily transitive on M . Let S be a subset of M such that $G \cdot S \subset S$. Each $X \in G$ defines a derivation of $C^\infty(S)$ that, since $\rho(X)$ is tangent to S , is compatible with the derivation $\rho(X): C^\infty(M) \rightarrow C^\infty(M)$ and the restriction mapping: $C^\infty(M) \rightarrow C^\infty(S)$. Formula 2.3 then defines, for $f \in C^\infty(S)$, a differential operator

$$\Delta_S: C^\infty(S) \rightarrow C^\infty(S).$$

It is clear that this is equivalent to saying that the operator Δ , when expressed in local coordinates about a point of S , has no component normal to S .

Now, fix a point $x_0 \in M$. Let $D = G \cdot x_0$, $L = L_{x_0}$, $M = M_{x_0}$, $L = K_{x_0}$. Let \bar{D} be the closure of D in M , and let $F = \bar{D} - D$ be its boundary. \bar{D} and F are invariant under the action of G . Suppose the basis (X_i) of G is chosen to satisfy 2.2 and so that (X_i) , $1 \leq i \leq m$, is a basis for L , (X_i) , $m+1 \leq i \leq n$, is a basis for M , and, for all such i , $(X_i, X_i) = A_i > 0$. Then, we see from 2.4 that:

Δ_D is a positive definite, second-order differential operator on the manifold D . Δ_F is a positive semi-definite operator on F .

DEFINITION 2.1. The *Bergman-Silov* boundary of D is the set B of $x \in F$ such that: $\Delta_F(f)(x) = 0$ for all $f \in C^\infty(F)$, i.e. B is the set of points where the coefficients of the operator Δ_F vanish identically.

It is clear from Lowdenslager's work [6] that B defined in this group-theoretic way has a close relation with the points of F at which functions in $C^\infty(\overline{D})$ that are harmonic with respect to Δ_D take their maximum value. Along this line, we now prove, using the same methods, the following generalizations of Lemma 1 of [6]:

PROPOSITION 2.2. *Suppose that $f \in C^\infty(\overline{D})$ satisfies: $\Delta(f) \geq 0$ and that F is compact.*

(a) *If there is an $\omega \in C^\infty(\overline{D})$ such that $\Delta(\omega)(x) > 0$ for all $x \in \overline{D} - B$, then f must take its maximum value at a point of B .*

(b) *If f has a strong relative maximum at $x \in F$, then $x \in B$.*

PROOF. If (a) were not true, there would be an $\epsilon > 0$ such that $f + \epsilon\omega$ takes a maximum value outside B . But, $\Delta(f + \epsilon\omega) > 0$ outside B , and $\Delta = \Delta_F$ on F , which contradicts the fact that the Hessian of $f + \epsilon\omega$ must be negative semi-definite at some point outside B .

Part (b) follows from the same argument on noting that such an ω can always be chosen locally about x if $x \in B$: $f + \epsilon\omega$ can be made to have a maximum value at a point arbitrarily close to x by choosing ϵ sufficiently small, since f takes a strong relative maximum at x . This again is a contradiction.

Recall that in Lowdenslager's situation, where M is Euclidean space, ω can be taken as a convex function.

3. The Poisson kernel. From now on, suppose that D and B are C^∞ manifolds on which a connected Lie group G acts. We do not assume that B lies on the boundary of D in any sense. We assume further that G has a quadratic form invariant under $\text{Ad } G$ that enables us to define invariant second-order linear operators on B and D , Δ_B and Δ_D . The problem to be discussed deals with conditions that at least some of the functions $f \in C^\infty(D)$ harmonic with respect to Δ_D can be expressed as integrals with respect to a "Poisson kernel" of functions on B .

Fix a point $x_0 \in D$ and let $L(=L_{x_0})$ be the isotropy group of G at x_0 . We suppose that:

(3.1) B has a C^∞ measure db (i.e., an everywhere nonzero differential form of maximal dimension) which is invariant under the action of K on B .

For $g \in G$ (resp. $X \in \mathfrak{G}$) let J_g (resp. J_X) be the Jacobian of T_g (resp. $\rho(X)$) with respect to db , i.e.

$$(3.2) \quad T_g^*(db) = J_g db,$$

$$(3.3) \quad \rho(X)(db) = J_X db.$$

(Recall that $\rho(X)$ denotes the operation of Lie derivation with respect to the vector-field on B defined by $X \in \mathfrak{G}$.)

We see from 3.2 and 3.3 that:

$$(3.4) \quad J_{\theta_1 \theta_2} = T_{\theta_2}^*(J_{\theta_1})J_{\theta_2} \quad \text{for } g, g_2 \in G.$$

$$(3.5) \quad J_{[X, Y]} = \rho(X)(J_Y) - \rho(Y)(J_X) \quad \text{for } X, Y \in \mathfrak{G}.$$

$$(3.6) \quad J_{\text{Ad}_g(X)} = T_g^*(J_X)J_g \quad \text{for } g \in G, X \in \mathfrak{G}.$$

A function $K(x, b) \in C^\infty(D, B)$ is said to be an *equivariant kernel* for $D \times B$ providing:

$$(3.7) \quad K(gx, b) = K(x, g^{-1}b)J_{g^{-1}}(b) \quad \text{for all } g \in G, b \in B, x \in D.$$

Let $C_0^\infty(B)$ be the subring of $C^\infty(B)$ of functions with compact support. A kernel $K(x, b)$ satisfying 3.7 defines a linear mapping $K: C_0^\infty(B) \rightarrow C^\infty(D)$ as follows:

$$(3.8) \quad K(f)(x) = \int K(x, b)f(b)db \quad \text{for } f \in C_0^\infty(B).$$

Note that:

$$(3.9) \quad T_g^*(K(f)) = K(T_g^*(f)) \quad \text{for } f \in C_0^\infty(B), g \in G.$$

Differentiating 3.7 with respect to g , we have:

$$(3.10) \quad \rho(X)_x(K(x, b)) = -\rho(X)_b(K(x, b)) - K(x, b)J_X(b) \\ \text{for all } X \in \mathfrak{G}, x \in D, b \in B.$$

$(\rho(X)_x$ and $\rho(X)_b$ denote partial Lie derivation with respect to the vector field defined by X on the x -space (i.e., D) and the b -space (i.e., B .) Applying $\rho(X)_x$ to 3.10 on both sides, and using 3.10 again, we have:

$$(3.11) \quad \rho(X)_x(\rho(X)_x(K(x, b))) \\ = \rho(X)_b(\rho(X)_b(K(x, b))) + 2\rho(X)_b(K(x, b))J_X(b) \\ + K(x, b)\rho(X)_b(J_X(b)) + K(x, b)(J_X(b))^2.$$

For $X \in \mathfrak{G}$, $x \in D$, define $F_{X,x} \in C^\infty(B)$ by:

$$(3.12) \quad F_{X,x}(b) = [2\rho(X)_b(K)J_X + K\rho(X)_b(J_X) + KJ_X^2](x, b),$$

i.e. everything but the first term on the right-hand side of 3.11. This is the "obstruction" to changing $\rho(X)_x\rho(X)_x$ into $\rho(X)_b\rho(X)_b$.

$$(3.13) \quad T_k^*(F_{X,x}) = F_{\text{Ad}_k(X), k^{-1}x}, \text{ using 2.1, 3.6 and 3.7.}$$

We are now ready to state the main result:

THEOREM 3.1. Suppose that, in addition to the conditions stated above,

(a) L is transitive on B and G is transitive on D ,

(b) Δ_B is identically zero,

(c) $K(x, b)$ satisfies 3.7 and

(d) $\int K(x, b) db = 1$ for all $x \in D$.

Then $K(x, b)$ is, for each b held fixed, a solution of the equation $\Delta_D = 0$. In particular, K maps all of $C_0^\infty(B)$ into solutions of the differential equation: $\Delta_D = 0$.

PROOF. With condition (b) we have, by 3.11,

$$(3.14) \quad \Delta_D K(x, b) = \sum_{i=1}^n A_i F_{X_i, x}(b).$$

Conditions (b) and (d) imply that

$$(3.15) \quad \int \sum_{i=1}^n A_i F_{X_i, x}(b) db = 0.$$

Notice, however, that for $k \in K$,

$$(3.16) \quad \sum_{i=1}^n A_i F_{Ad k(X_i), x_0}(kb) = \sum_{i=1}^n A_i F_{X_i, x_0}(b).$$

Referring to 3.12, 3.6, and 2.1, we see that:

$$(3.17) \quad \sum_{i=1}^n A_i F_{Ad k(X_i), x_0} = \sum_{i=1}^n A_i F_{X_i, x_0},$$

i.e., using 3.16 and condition (a), the integrand in 3.15 is constant. Hence:

$$(3.18) \quad \sum_{i=1}^n A_i F_{X_i, x_0}(b) = 0 \quad \text{for all } b \in B.$$

For each $b \in B$, let $K_b \in C^\infty(D)$ be the function:

$$K_b(x) = K(x, b) \quad \text{for } x \in D.$$

Then, by 3.14 and 3.18,

$$(3.19) \quad \Delta_D(K_b)(x_0) = 0 \quad \text{for all } b \in B.$$

To finish the proof, we must show that this holds for all points of D . Suppose that $x = g \cdot x_0 \in D$ for some $g \in G$.

$$\begin{aligned} \Delta_D(K_b)(x) &= T_g^*(\Delta_D K_b)(x_0), \\ &= \Delta_D(T_g^*(K_b))(x_0), \text{ using invariance of } \Delta_D \text{ under } G, \\ &= \Delta_D(K_{g^{-1}b} J_{g^{-1}}(b))(x_0), \text{ by 3.7,} \\ &= \Delta_D(K_{g^{-1}b})(x_0) J_{g^{-1}}(b) = 0, \text{ by 3.19.} \end{aligned}$$

This completes the proof of Theorem 3.1.

PROPOSITION 3.2. *Suppose that G acts on manifolds D and B , that B is compact, $L \subset G$ is the isotropy sub group of G at $x_0 \in D$, that K is transitive on B , and that the isotropy subgroup L_x of G at each $x \in D$ is compact.*

Then, there is a C^∞ measure db on B and a function $K(x, b) \in C^\infty(D, B)$ satisfying 3.7 and the condition (d) of Theorem 3.1.

PROOF. Since L_x and B are compact, there is a C^∞ measure $d_x b$ on B invariant under L_x such that

$$\int_B d_x b = 1 \quad \text{for each } x \in D.$$

Set: $db = d_{x_0} b$, and $d_x b = K(x, b) db$.

One sees easily that $K(x, b)$ satisfies the required conditions. Note also that:

$$(3.20) \quad K(x, b) = J_{g^{-1}}(b) \text{ if } x = g \cdot x_0.$$

REMARKS. (a) **Boundary behavior.** For $X \in \mathfrak{G}$, suppose that there is a subset $A \subset B$ of measure zero and a point $b_0 \in B$ such that:

$$(3.21) \quad \text{For all } b \in B - A, \quad \lim_{t \rightarrow \infty} \text{Exp}(tX) \cdot b = b_0.$$

Following Lowdenslager [6] we note then that:

$$(3.22) \quad \lim_{t \rightarrow \infty} \int K(\text{Exp}(tX) \cdot x_0, b) f(b) db = f(b_0) \text{ (use 3.20) for all } f \in C_0^\infty(B),$$

i.e. the function in D defined by the Poisson formula has the correct boundary behavior along the ray starting from x_0 determined by X . One can prove that the following conditions imply 3.21.

- (3.23) (a) B is compact.
 (b) There is a Riemannian metric on B and a $f \in C^\infty(B)$ such that $\rho(X)$ on B is the gradient field of f .
 (c) f has only one relative maximum, at b_0 .
 (d) Let $N \subset B$ be any connected component of the set of other critical points of f . We require that there be a neighborhood U of N and a submanifold S of U of lower dimension containing N such that, for $b \in U$, the curve $t \rightarrow \text{Exp } tX \cdot b$ approaches N as $t \rightarrow \infty$ if and only if $b \in S$. (S is a *stable manifold* for the vector field $\rho(X)$ near its critical set).

The point is that, at least for the case where D is a Cartan domain

and M is its compactification, it can be shown that these conditions follow from rather general differential-geometric arguments. We will come back to this point of view in a later work.

(b) **The general setting of Theorem 3.1.** Theorem 3.1 involves a condition for commutativity of equivariant differential and integral operators. For other sufficient conditions for this sort of phenomenon, see [8].

(c) **Application to the Cartan domains.** Suppose that D is a Cartan domain and that M is the compact Kähler manifold in which D is equivariantly embedded as an open set. Then, as above, $D = G/L$, where G is a non-compact, simple Lie group, L is a maximal connected compact subgroup of G , $M = G_u/L$, where G_u is a compact real form of the complex simple Lie group determined by G . G/L is then a Riemannian symmetric space, and G admits a decomposition $L \oplus M$, with $[M, M] \subset L$. Let H be a maximal abelian subalgebra of M . Let W be a Weyl chamber of H . Define $B \subset \bar{D} - D$ as in definition 2.1. It follows from the results of [4] that $\lim_{t \rightarrow \infty} \exp(Xt) \cdot x_0 \in B$ for $X \in W$, and for X lying on the boundary of W , $\lim_{t \rightarrow \infty} \exp Xt \cdot x_0 \notin B$. R. Bott and A. Koranyi have shown (to appear) that there is a $b_0 \in B$ such that: $\lim_{t \rightarrow \infty} \exp Xt \cdot x_0 = b_0$ for all $X \in W$, and that L acts transitively on the orbit of G at b_0 . (We can show this also with the geometric method described in remark (a) above.) It follows from these facts that G and L both act transitively on B .

All of these facts are due to D. Lowdenslager, ([6], together with some unpublished work) in the classical cases. However, it is useful for the sake of possible generalizations to have general proofs of these results.

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