

## REFERENCES

1. M. L. Cartwright, *Some decomposition theorems for certain invariant continua and their minimal sets*, Fund. Math. vol. 48 (1960) pp. 229–250.
2. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloquium Publications, Vol. 36, 1955.
3. W. H. Gottschalk, *Minimal sets: An introduction to topological dynamics*, Bull. Amer. Math. Soc. vol. 64 (1958) pp. 336–351.

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 THE KRONECKER PRODUCT OF GRAPHS

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**Introduction.** This note considers a graph product derived from the Kronecker product of matrices. Some indication of the geometrical nature of this product is given and a theorem stating necessary and sufficient conditions for a graph product to be connected is proved. The matrix analogue of the above result is also stated.

I. A convenient representation for a finite undirected graph [1]  $G$  is an adjacency matrix. If the vertex set of  $G$  is  $\{p_i\}$ ,  $i = 1, \dots, n$ , then an adjacency matrix of  $G$  is an  $n \times n$  matrix  $(a_{ij})$  with  $a_{ij}$  equal to the number of lines (paths of length one) joining  $p_i$  to  $p_j$ . A given graph is then represented by an equivalence class of matrices  $\bar{A} = \{P_i A P_i^{-1} \mid \text{for all permutation matrices } P_i \text{ of order equal to the order of } A\}$ . Each element of  $\bar{A}$  corresponds to a different ordering of the vertices of  $G$ . It is clear that for each such class of adjacency matrices there corresponds a unique class of isomorphic graphs.

From this point on "graph" will mean a finite undirected graph with no loops. Such a graph has an adjacency matrix  $(a_{ij})$  whose entries are non-negative integers such that  $a_{ij} = a_{ji}$  and  $a_{ii} = 0$ . We also use the following notation:  $o(G)$  is the number of vertices of  $G$  and is called the *order* of  $G$ ,  $p_1 \rightarrow p_k$  is a chain in  $G$  from vertex  $p_1$  to vertex  $p_k$ , and  $n(p_1 \rightarrow p_k)$  is the number of lines (not necessarily distinct) in  $p_1 \rightarrow p_k$ .

If  $A$  and  $B$  are two adjacency matrices and  $A \circ B$  is some matrix product which is also an adjacency matrix then this matrix operation

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induces a composition of the graphs represented by  $A$  and  $B$ . In order that a given matrix product induce an appropriate graph product the following conditions must be satisfied.

- (1)  $A \circ B$  should be an adjacency matrix of some graph.
- (2) For any permutation matrices  $P_1$  and  $P_2$  of appropriate orders there exists a permutation matrix  $P$  such that:

$$P(A \circ B)P^{-1} = (P_1AP_1^{-1}) \circ (P_2BP_2^{-1}).$$

That is, the matrices  $A \circ B$  and  $(P_1AP_1^{-1}) \circ (P_2BP_2^{-1})$  are equivalent in the sense defined above.

Two other properties are desirable.

- (3)  $A \circ B = B \circ A$  and
- (4)  $(A \circ B) \circ C = A \circ (B \circ C)$ .

One matrix operation which satisfies all of the above requirements is the Kronecker product. Hence we may define the Kronecker product of two graphs as follows:

**DEFINITION 1.** Let  $A$  and  $B$  be adjacency matrices of the graphs  $G$  and  $H$  respectively. The *Kronecker product*,  $G \otimes H$ , is that graph with adjacency matrix  $A \otimes B$ , the Kronecker product of  $A$  and  $B$ .

Notice that condition (2) above guarantees that regardless of the choice of  $A$  and  $B$  the graph associated with  $A \otimes B$  will be the same up to isomorphism.

The Kronecker product of two graphs can also be represented in a somewhat more direct manner.

**DEFINITION 1'.** Let the vertex sets of the graphs  $G$  and  $H$  be  $\{p_i\}$ ,  $i=1, \dots, n$ , and  $\{q_j\}$ ,  $j=1, \dots, m$ , then  $G \otimes H$  is a graph with vertex set  $\{p_iq_j \mid p_iq_j = q_jp_i \text{ for } i=1, \dots, n, j=1, \dots, m\}$  such that the adjacency number of the pair  $p_iq_j, p_kq_l$  is the product of the adjacency numbers of  $p_i, p_k$  in  $G$  and  $q_j, q_l$  in  $H$ .

It is easy to show that Definition 1 and Definition 1' are equivalent.

As an illustration of the geometrical properties of the Kronecker product consider the product of two stars. A star is a graph with one vertex singly adjacent to all other vertices and no other adjacencies. Clearly such a graph has an adjacency matrix  $(a_{ij})$  with  $a_{ij} = a_{ji} = 1$  for  $i, j \neq 1$  and  $a_{ij} = 0$  otherwise. If  $G$  and  $H$  are stars of orders  $a$  and  $b$  respectively then  $G \otimes H$  is a graph with adjacency matrix

$$\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix},$$

where  $C$  is an adjacency matrix of a star of order  $(a-1)(b-1)+1$  and  $D$  is an adjacency matrix of a graph whose vertex set can be

partitioned into two sets containing  $a-1$  and  $b-1$  vertices respectively such that each vertex is adjacent to every vertex not in its set and to no others. This can be shown either by applying an appropriate sequence of permutations to the rows and columns of the Kronecker product of adjacency matrices or by listing the vertices of  $G \otimes H$  and deriving the adjacency relations by the criterion of Definition 1'. The graph defined by  $D$  can also be described as a "star" with  $a-1$  "end-points" and  $b-1$  "centers." For example, let  $G$  and  $H$  be stars of orders 6 and 4 respectively. Then

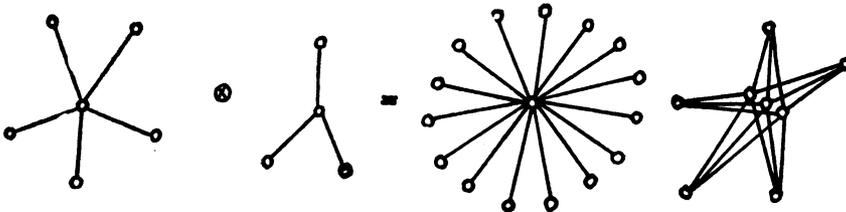


FIGURE 1

Notice that the product of these two graphs is a disconnected graph. We now examine the question as to when the Kronecker product of two graphs is connected.

**LEMMA 1.** *Let  $G$  and  $H$  be graphs with vertex sets  $\{p_i\}$  and  $\{q_j\}$  respectively. If  $p_1q_1$  and  $p_2q_2$  are two vertices of  $G \otimes H$  and if there exist  $p_1 \rightarrow p_2$  in  $G$  and  $q_1 \rightarrow q_2$  in  $H$  such that  $n(p_1 \rightarrow p_2) + n(q_1 \rightarrow q_2)$  is even then there exists  $p_1q_1 \rightarrow p_2q_2$  in  $G \otimes H$ .*

**PROOF.** Suppose  $n(p_1 \rightarrow p_2) \leq n(q_1 \rightarrow q_2)$ . Then there exists  $p_1q_1 \rightarrow p_2q_k$  in  $G \otimes H$  for some  $q_k$  in  $H$  and the path  $q_k \rightarrow q_2$  has an even number of lines. If  $p_i$  is a vertex adjacent to  $p_2$  in  $G$  and  $(q_k, q_{k+1}, \dots, q_2) = q_k \rightarrow q_2$  then the path  $(p_2q_k, p_1q_{k+1}, p_2q_{k+2}, \dots, p_2q_2)$  connects  $p_2q_k$  to  $p_2q_2$ . Hence the composite of  $p_1q_1 \rightarrow p_2q_k$  and  $p_2q_k \rightarrow p_2q_2$  connects  $p_1q_1$  to  $p_2q_2$  in  $G \otimes H$ .

A cycle of a graph is called *odd* if it contains an odd number of vertices.

**THEOREM 1.** *Let  $G$  and  $H$  be connected graphs.  $G \otimes H$  is connected if and only if either  $G$  or  $H$  contains an odd cycle.*

**PROOF.** Let the vertex sets of  $G$  and  $H$  be  $\{p_i\}$  and  $\{q_j\}$  respectively. Let  $q_1$  and  $q_r$  be adjacent vertices in  $H$  and assume that  $G \otimes H$  is connected. Hence there exists  $(p_1q_1 \rightarrow p_1q_r) = (p_1q_1, p_2q_2, \dots, p_uq_u, p_1q_r)$  in  $G \otimes H$ . Associate with each vertex in this chain a pair of non-

negative integers,  $p_i q_j \sim (\alpha_i, \beta_j)$  with  $\alpha_i$  equal to the length of a shortest chain from  $p_i$  to  $p_1$  in  $G$  and  $\beta_j$  the length of a shortest path from  $q_j$  to  $q_r$  in  $H$ . Clearly  $p_1 q_1 \sim (0, 1)$  and  $p_u q_v \sim (1, 1)$ . If  $p_u q_x$  and  $p_v q_z$  are adjacent vertices in the chain above then  $|\alpha_v - \alpha_w| \leq 1$  and  $|\beta_x - \beta_z| \leq 1$ . For otherwise the minimality of the chains would be violated. If at each step in the path both components experience a nonzero change then the sums  $\alpha_i + \beta_j$  would have the same parity. But since  $p_1 q_1 \sim (0, 1)$  and  $p_u q_v \sim (1, 1)$  it follows that at one step in the chain either  $\alpha_v - \alpha_w = 0$  or  $\beta_x - \beta_z = 0$ . Let  $\beta_x - \beta_z = 0$ . Then there are two minimal chains  $q_x \rightarrow q_r$  and  $q_z \rightarrow q_r$ . Let  $q_m$  be the first vertex common to both of these chains. Now  $n(q_x \rightarrow q_m) = n(q_z \rightarrow q_m)$ . For otherwise the original chain would not be minimal. Hence the cycle  $(q_x \rightarrow q_m, q_m \rightarrow q_z, q_z)$  is odd. The same argument applies if  $\alpha_v = \alpha_w$ . Therefore either  $G$  or  $H$  contains an odd cycle.

Conversely, let  $G$  contain an odd cycle. Let  $p_1 q_1$  and  $p_2 q_2$  be arbitrary vertices of  $G \otimes H$ . Since  $G$  and  $H$  are connected, there are chains  $p_1 \rightarrow p_2$  and  $q_1 \rightarrow q_2$  in  $G$  and  $H$  respectively. If  $n(p_1 \rightarrow p_2) + n(q_1 \rightarrow q_2)$  is even then from Lemma 1 it follows that  $p_1 q_1$  is connected to  $p_2 q_2$  in  $G \otimes H$ . Suppose  $n(p_1 \rightarrow p_2) + n(q_1 \rightarrow q_2)$  is odd. Let  $p_1 \rightarrow p_1$  be a chain which includes exactly one odd cycle of  $G$ . Then  $n(p_1 \rightarrow p_1)$  is odd and since  $n(p_1 \rightarrow p_1, p_1 \rightarrow p_2) = n(p_1 \rightarrow p_1) + n(p_1 \rightarrow p_2)$ , it follows that  $n(p_1 \rightarrow p_1, p_1 \rightarrow p_2) + n(q_1 \rightarrow q_2)$  is even. Hence  $p_1 q_1$  is connected to  $p_2 q_2$  in  $G \otimes H$ . A similar argument holds if we assume that  $H$  contains an odd cycle.

**COROLLARY.** *If  $G$  and  $H$  are connected graphs with no odd cycles then  $G \otimes H$  has exactly two connected components.*

**PROOF.** If  $q_1$  is adjacent to  $q_2$  in  $H$  then  $p_1 q_1$  and  $p_1 q_2$  are not connected. For if they were it would follow that either  $G$  or  $H$  had an odd cycle, contrary to hypothesis. Let  $p_k q_k$  be an arbitrary vertex of  $G \otimes H$ . It will suffice to show that  $p_k q_k$  is connected either to  $p_1 q_1$  or  $p_1 q_2$ . Consider chains  $p_k \rightarrow p_1$  and  $q_k \rightarrow q_1$ . If  $n(p_k \rightarrow p_1) + n(q_k \rightarrow q_1)$  is even then by Lemma 1  $p_k q_k$  is connected to  $p_1 q_1$ . Otherwise consider the chain  $(q_k \rightarrow q_1, q_2)$ .  $n(q_k \rightarrow q_1, q_2) = n(q_k \rightarrow q_1) + 1$  and hence  $n(p_k \rightarrow p_1) + n(q_k \rightarrow q_1, q_2)$  is even. It therefore follows from Lemma 1 that  $p_k q_k$  is connected to  $p_1 q_2$ . This completes the proof of the Corollary.

II. If  $G$  is a disconnected graph then  $G$  may be represented by a decomposed matrix

$$A = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix},$$

with  $C$  and  $D$  square matrices. We will say that an adjacency matrix  $M$  is *decomposable* if there exists a permutation matrix  $P$  such that  $PMP^{-1}$  is decomposed. Hence Theorem 1 gives a necessary and sufficient condition that the Kronecker product of two adjacency matrices be indecomposable. If this condition, the existence of an odd cycle, can be characterized matrix-wise then Theorem 1 can be restated purely in matrix terms. We now proceed to do this.

LEMMA 2. *A graph  $G$  contains an odd cycle if and only if it possesses an adjacency matrix*

$$A = \begin{pmatrix} C & * \\ * & * \end{pmatrix}$$

*with  $C$  an odd order matrix each row of which contains exactly two non-zero entries.*

PROOF. Let  $G$  be a graph with vertex set  $\{p_i\}$  which contains an odd cycle. Let  $p_i \rightarrow p_i$  be an odd cycle of minimal length in  $G$ . Re-number the vertices of  $G$  choosing the vertices of  $p_i \rightarrow p_i$  as the first  $k$  and the rest arbitrarily. Let  $A$  be the associated adjacency matrix. Then

$$A = \begin{pmatrix} C & * \\ * & * \end{pmatrix}$$

with  $C$  of order  $k$ , and every row of  $C$  contains at least two nonzero entries. If any row were to contain more than two nonzero entries then there would exist an odd cycle composed of the vertices represented by  $C$  which contained less than  $k$  vertices, contradicting the minimality of  $k$ . Hence every row contains exactly two nonzero entries.

Conversely, if  $A$  is an adjacency matrix of the form

$$\begin{pmatrix} C & * \\ * & * \end{pmatrix}$$

with  $C$  an odd order matrix containing exactly two nonzero entries in every row then  $C$  is an adjacency matrix of a graph containing an odd cycle. For if every vertex of a graph is adjacent to exactly two others then, neglecting multiple connections, the graph is composed of a set of disconnected cycles. But since  $C$  has odd order one of these cycles must be odd.

Hence we may restate Theorem 1.

**THEOREM 1'.** *Let  $A$  and  $B$  be two indecomposable adjacency matrices. Then the Kronecker product  $A \otimes B$  is indecomposable if and only if there exists either  $P_1$  or  $P_2$ , permutation matrices, such that*

$$P_1 A P_1^{-1} = \begin{pmatrix} C & * \\ * & * \end{pmatrix}$$

*with  $C$  an odd order matrix containing exactly two nonzero entries in every row or*

$$P_2 B P_2^{-1} = \begin{pmatrix} D & * \\ * & * \end{pmatrix}$$

*with  $D$  similarly defined.*

#### REFERENCES

1. D. König, *Theorie der endlichen und unendlichen graphen*, Akademie Verlag M.B.H., Leipzig, 1936.

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