OPERATIONS ON TOR AND EXT

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1. The operation of a group on the homology and cohomology of a normal subgroup, or of a Lie algebra on the homology and cohomology of an ideal, has been defined and used for some time. However, a systematic functorial treatment of such operations has never been given explicitly, and it is the purpose of this note to supply such a treatment in a suitably general form.

Let K be a commutative ring with identity, and let R be a K-algebra with identity. Let α be a K-algebra automorphism of R, and let τ be an α-derivation of R, i.e., a K-linear endomorphism such that, for all x and y in R, $\tau(xy)=\alpha(x)\tau(y)+\tau(x)\alpha(y)$. We are interested in the maps of Tor and Ext that are naturally connected with α and τ. Much of the formal development can be given simultaneously for automorphisms and derivations by considering suitable pairs of module maps.

Let U and V be unitary left R-modules. By an (α, τ)-pair of maps $U\to V$ we mean a pair (p, q) of K-module homomorphisms of U into V such that, for all $r\in R$ and all $u\in U$,

$$p(r\cdot u) = \alpha(r)\cdot p(u), \quad \text{and} \quad q(r\cdot u) = \alpha(r)\cdot q(u) + \tau(r)\cdot p(u).$$

There is a composition of such pairs, as follows. Suppose that $(p_1, q_1)\colon V\to W$ is an (α_1, τ_1)-pair. Then one verifies directly that (p_1p, p_1q+q_1p) is an $(\alpha_1\alpha, \alpha_1\tau+\tau_1\alpha)$-pair of maps $U\to W$, and we define the composite $(p_1, q_1)(p, q)$ to be this pair. Exactly analogous definitions are in force for right modules.

Suppose we are given an (α, τ)-pair $(p_1, q_1)\colon U\to U'$, for right modules, and an (α, τ)-pair $(p_2, q_2)\colon V\to V'$, for left modules. Then it is easily verified that there is a pair (P, Q) of K-module homomorphisms $U\otimes_R V\to U'\otimes_R V'$ such that

$$P(u \otimes v) = p_1(u) \otimes p_2(v), \text{and } Q(u \otimes v) = p_1(u) \otimes q_2(v) + q_1(u) \otimes p_2(v).$$

We call (P, Q) the pair induced by (p_1, q_1) and (p_2, q_2). Composites of such induced pairs are defined as above: $(P, Q)(P', Q') =(PP', PQ'+QP')$. One verifies directly that this composite is the pair induced by the composites of the maps (p, q).

Now let U, U', V, V' be left R-modules, and suppose we are given an (α, τ)-pair $(p, q)\colon V\to V'$ and an $(\alpha^{-1}, -\alpha^{-1}\tau\alpha^{-1})$-pair $(p', q')\colon U'\to U$. Then there is a pair (P, Q) of K-module homomorphisms $\mathrm{Hom}_R(U, V)\to\mathrm{Hom}_R(U', V')$ such that

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$$P(h) = php', \quad \text{and} \quad Q(h) = phq' + qhp'.$$

This pair (P, Q) is called the pair induced by (p, q) and (p', q'). Composites are defined as before, and they coincide with the pairs induced by the composites of the pairs (p, q).

We say that an R-module A is (α, τ)-projective if, for any R-module epimorphism ϕ: $U \to V$ and any (α, τ)-pair (p, q): $A \to V$, there is an (α, τ)-pair (p', q'): $A \to U$ such that $\phi p' = p$ and $\phi q' = q$. We say that A is (α, τ)-injective if every (α, τ)-pair (p, q): $U \to A$ can be extended to an (α, τ)-pair (p', q'): $W \to A$, for every R-module W containing U.

PROPOSITION 1. *Every projective R-module is (α, τ)-projective.*

PROOF. It is evidently sufficient to prove that R is (α, τ)-projective when regarded as a left (or right) R-module. Let ϕ be an R-module epimorphism $U \to V$, and let (p, q) be an (α, τ)-pair $R \to V$. Choose elements u_p and u_q in U such that $\phi(u_p) = p(1)$ and $\phi(u_q) = q(1)$. Now define $p'(r) = \alpha(r) \cdot u_p$ and $q'(r) = \alpha(r) \cdot u_q + \tau(r) \cdot u_p$. One verifies directly that (p', q') satisfies the requirements of Proposition 1.

PROPOSITION 2. *Every injective R-module is (α, τ)-injective.*

PROOF. As is well known, every injective R-module is a direct R-module summand of an R-module of the type $\mathrm{Hom}_Z(R, B)$, where Z is the ring of the rational integers and B is a Z-injective Z-module, the R-module structure being given by $(r \cdot h)(r_1) = h(r_1 r)$. Hence it suffices to show that $\mathrm{Hom}_Z(R, B)$ is (α, τ)-injective.

Let W be an R-module, U a submodule of W, (p, q) an (α, τ)-pair $U \to \mathrm{Hom}_Z(R, B)$. Define the pair (p^*, q^*): $U \to B$ by setting $p^*(u) = p(u)(1)$ and $q^*(u) = q(u)(1)$. Then p^* and q^* extend to Z-homomorphisms p_1 and q_1 (respectively) of W into B. Define the maps p' and q' of W into $\mathrm{Hom}_Z(R, B)$ by $p'(w)(r) = p_1(\alpha^{-1}(r) \cdot w)$ and $q'(w)(r) = q_1(\alpha^{-1}(r) \cdot w) - p_1(\alpha^{-1}\tau\alpha^{-1}(r) \cdot w)$. One verifies directly that (p', q') is an (α, τ)-pair $W \to \mathrm{Hom}_Z(R, B)$ extending (p, q). This completes the proof of Proposition 2.

Now let (p, q) be an (α, τ)-pair of maps $U \to V$ of left R-modules. Let X be a projective R-complex ending at U, and let Y be an acyclic R-complex ending at V. Exactly as in the ordinary theory of R-complexes, we show, using Proposition 1, that (p, q) can be extended to an (α, τ)-pair of homogeneous K-complex maps of degree 0 of X into Y. Moreover, if (p', q') and (p'', q'') are any two such extensions of (p, q), there exists an (α, τ)-pair (p_1, q_1): $X \to Y$ that is homogeneous of degree 1 and such that, if d denotes the boundary map in X or Y,

$$dp_1 + p_1 d = p' - p'', \quad \text{and} \quad dq_1 + q_1 d = q' - q''.$$

Dually, if X is an acyclic R-complex beginning at U and Y is an injective R-complex beginning at V, (p, q) can be extended as above, with the same type of uniqueness.

It is clear from this that the induced pairs (P, Q) of maps on Hom_R or $\otimes_R$ extend via resolutions to pairs of maps on Ext_R and Tor^R, respectively. The independence of these pairs on the choice of resolutions and extensions to resolutions of (p, q) follows immediately from the uniqueness up to homotopy pairs of the pairs of maps of resolutions.

2. The simplest general example of (α, τ)-pairs is the following. Let a be an invertible element of R and let t be an arbitrary element of R. Define the *inner* automorphism α of R by $\alpha(r) = ara^{-1}$ and define the *inner* α-derivation τ of R by $\tau(r) = a(tr - rt)a^{-1}$. Let U be a right R-module, V a left R-module. We define *canonical inner* (α, τ)-pairs $(p_1, q_1)\colon U \to U$ and $(p_2, q_2)\colon V \to V$ by

$$p_1(u) = u \cdot a^{-1}, \quad q_1(u) = - u \cdot (ta^{-1}), \quad p_2(v) = a \cdot v, \quad q_2(v) = (at) \cdot v.$$

The induced pair (P, Q) on $U \otimes_R V$ is immediately seen to be the pair $(1, 0)$, where 1 is the identity map. Clearly, the pairs (p_i, q_i) can be extended to projective resolutions of U and V by the same formulas. Hence the induced pair on $\mathrm{Tor}_R(U, V)$ is $(1, 0)$.

Now let U and V be left R-modules. Let $(p, q)\colon V \to V$ be defined as (p_2, q_2) was defined above, and let $(p', q')\colon U \to U$ be defined similarly for $(\alpha^{-1}, -\alpha^{-1}\tau\alpha^{-1})$. Since α^{-1} is the inner automorphism effected by a^{-1} and $-\alpha^{-1}\tau\alpha^{-1}$ is the inner α^{-1}-derivation effected by $-ata^{-1}$, this means that we define

$$p'(u) = a^{-1} \cdot u, \quad \text{and} \quad q'(u) = - (a^{-1}(ata^{-1})) \cdot u = - (ta^{-1}) \cdot u.$$

The induced pair (P, Q) on $\mathrm{Hom}_R(U, V)$ is immediately seen to be $(1, 0)$. As before, it follows that the induced pair on $\mathrm{Ext}_R(U, V)$ is also $(1, 0)$.

3. Let G be a group, and let ρ be a representation of G in the automorphism group of the K-algebra R. Let U be a left R-module, and let p_U be a representation of G in the group of the K-automorphisms of U such that, for all $g \in G$, $u \in U$, $r \in R$, we have

$$p_U(g)(r \cdot u) = \rho(g)(r) \cdot p_U(g)(u).$$

We also consider analogous representations p_U for right R-modules U. Then each $(p_U(g), 0)$ is a $(\rho(g), 0)$-pair. It is easily seen from what we have said above concerning resolutions that the standard procedure, carried out with representations p_U and p_V in R-modules U and V, yields representations of G in the K-automorphism groups of $\mathrm{Ext}_R(U, V)$ and $\mathrm{Tor}^R(U, V)$.

Similarly, let $\mathfrak{G}$ be a K-Lie algebra, and let ρ be a representation of $\mathfrak{G}$ in the Lie algebra of all K-derivations of R. For a left R-module U, we consider a representation q_U of $\mathfrak{G}$ in the Lie algebra of the K-endomorphisms of U such that, for all $\zeta \in \mathfrak{G}$, $u \in U$, $r \in R$, we have

$$q_U(r \cdot u) = \rho(\zeta)(r) \cdot u + r \cdot q_U(\zeta)(u).$$

Then each $(1, q_U(\zeta))$ is a $(1, \rho(\zeta))$-pair. We also consider analogous representations q_U in right R-modules U. Our standard procedure, carried out with representations q_U and q_V in R-modules U and V, yields representations of $\mathfrak{G}$ in the Lie algebras of the K-endomorphisms of $\mathrm{Ext}_R(U, V)$ and $\mathrm{Tor}^R(U, V)$. In manipulating the resolutions, it must be observed that if $(1, q_i)$ is a $(1, \rho(\zeta_i))$-pair, for $i=1, 2$, then $(1, [q_1, q_2])$ is a $(1, \rho([\zeta_1, \zeta_2]))$-pair.

Now let G be a group, H a normal subgroup of G. Let $R=K[H]$, the group algebra of H over K. For $g \in G$, let $\rho(g)$ be the automorphism of R given by $\rho(g)(r)=grg^{-1}$. We consider $K[G]$-modules, regarding them also as $K[H]$-modules in the natural fashion. If U is a left $K[G]$-module we define, for each $g \in G$, $p_U(g)$ to be the automorphism of U corresponding to g in the left $K[G]$-module structure of U. If U is a right $K[G]$-module, $p_U(g)$ is defined to be the automorphism corresponding to g^{-1}. With these definitions, our above representations of G in the automorphism groups of $\mathrm{Ext}_R(U, V)$ and $\mathrm{Tor}^R(U, V)$ become the usual ones. Exactly analogous considerations apply to the case of a Lie algebra $\mathfrak{G}$ and an ideal $\mathfrak{H}$ of $\mathfrak{G}$. Note that it follows immediately from our above discussion of inner automorphisms and derivations that the restrictions to H and $\mathfrak{H}$ of our representations of G and $\mathfrak{G}$ on Ext_R and Tor^R are trivial.

4. Almost all of our above considerations are involved in the following situation, which we sketch for the sake of illustration. Let G be a real or complex analytic group, let H be a normal analytic subgroup of G, and let $\mathfrak{G}$, $\mathfrak{H}$ denote the Lie algebras of G, H, respectively. Let R be the universal enveloping algebra of $\mathfrak{H}$. Let K stand for the field of the real or complex numbers, according to whether G is real or complex. We consider locally finite analytic representation spaces (or anti-representation spaces) for G. Let U be such a representation space, and let p_U denote the representation of G on U. Then the differential $\dot{p_U}$ of p_U defines the structure of a (locally finite) representation space for $\mathfrak{G}$ on U, and hence (by restriction to $\mathfrak{H}$ and canonical extension to R) that of a left R-module. Similarly, if p_U is an anti-representation, it induces the structure of an anti-representation space for $\mathfrak{G}$ and hence that of a right R-module on U.

On the other hand, the adjoint representation α of G on $\mathfrak{H}$ induces a locally finite analytic representation of G by automorphisms of R.

Hence its differential $\alpha^{\cdot}$ induces a locally finite representation of $\mathfrak{G}$ by derivations of R. The standard facts on analytic representations of analytic groups and their differentials show that, for each $g \in G$ and each $\zeta \in \mathfrak{G}$, $(p_U(g), p_U(g)p_U^{\cdot}(\zeta))$ is an $(\alpha(g), \alpha(g)\alpha^{\cdot}(\zeta))$-pair.

If P is any locally finite analytic representation space for G then the tensor product representation of G on $R \otimes_K P$ makes $R \otimes_K P$ into a locally finite analytic representation space for G. Moreover, the map $R \otimes_K P \to P$ sending $r \otimes p$ onto $r \cdot p$ is a G-epimorphism, as follows from the standard results on the analytic representations of G. It follows that a locally finite analytic representation p_U of G on U can be extended to a locally finite analytic representation of G on the projective resolution of U that is obtained by iterating the canonical epimorphisms $R \otimes_K P \to P$ (starting with $P = U$, then taking P to be the kernel of the map $R \otimes_K U \to U$, etc.). By considering the induced action of G on the tensor product of such resolutions of two locally finite analytic representation (anti-representation) spaces U and V for G, one sees easily that $\mathrm{Tor}^R(U, V)$ becomes a locally finite analytic representation space for G. By examining the action of $\exp(t\zeta)$ (where $\zeta \in \mathfrak{G}$ and t is a variable in K) one verifies easily that the representation of $\mathfrak{G}$ on $\mathrm{Tor}^R(U, V)$ is the differential of the representation of G on $\mathrm{Tor}^R(U, V)$.

Now let U be a finite dimensional analytic representation space for G and let V be a locally finite analytic representation space for G. As is well known, there is a natural functorial isomorphism between $\mathrm{Ext}_R(U, V)$ and $\mathrm{Ext}_R(K, \mathrm{Hom}_K(U, V))$. Hence we may compute the action of G and $\mathfrak{G}$ on $\mathrm{Ext}_R(U, V)$ from an R-projective resolution X of K. Taking for X the usual resolution $X = R \otimes_K E$, where E is the exterior algebra constructed over $\mathfrak{H}$, we have the structure of a locally finite analytic representation space for G on X, which is induced in the natural fashion by the adjoint representation α. Moreover, since U and E are finite dimensional, the induced representation of G on $\mathrm{Hom}_R(X, \mathrm{Hom}_K(U, V)) = \mathrm{Hom}_K(E, \mathrm{Hom}_K(U, V))$ is locally finite analytic. It follows that the induced representation of G on $\mathrm{Ext}_R(U, V)$ is locally finite analytic. By examining the action of $\exp(t\zeta)$ on $\mathrm{Hom}_R(X, \mathrm{Hom}_K(U, V))$ we see easily that the representation of $\mathfrak{G}$ on $\mathrm{Ext}_R(U, V)$ is the differential of the representation of G.

The same treatment can be given almost word for word in the case of rational representations of linear algebraic groups over a field of characteristic 0.

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