PROOF OF A CONJECTURE ABOUT MEASURABLE SETS ON THE REAL LINE

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In [1, p. 243] the author formulated a conjecture concerning Lebesgue-measurable sets on the real line; this conjecture was connected with an analysis of the structure of a class of translation-invariant function spaces. The purpose of this note is to give a proof of that conjecture. The proof is based on an idea of E. Calabi. The research was carried out while the author held a fellowship granted by the John Simon Guggenheim Memorial Foundation.

Let R denote the real line and let μ be Lebesgue measure on R. The sign $\setminus$ is used to denote set-theoretical differences. If F, $F' \subset R$ are μ-measurable, we say that F *is essentially contained in* F' if $\mu(F \setminus F') = 0$.

THEOREM. *If $E \subset R$ is μ-measurable and $\mu(E) > 0$, there exists a closed, bounded set $E' \subset R$ with $\mu(E') > 0$, such that E is not essentially contained in any finite union of translates of E'.*

PROOF. For every measurable set $F \subset R$ with $\mu(F) > 0$ and for every real $s > 0$ we define

$$\omega(s; F) = s^{-1} \cdot \inf_{t \in R} \mu(\{(t + [0, s]) \setminus F\}).$$

Obviously, $0 \leq \omega(s; F) \leq 1$. Since $\mu(F) > 0$, there exists a point $t_0 \in F$ of density 1 with respect to F, and therefore

$$\lim_{s \to 0} s^{-1} \cdot \mu(\{(t_0 + [0, s]) \setminus F\}) = 0;$$

a fortiori we have

$$\lim_{s \to 0} \omega(s; F) = 0. \tag{1}$$

Let E be the given set. On account of (1) we may construct a sequence of positive integers $\{m(n)\}$ such that the following two conditions are satisfied for all $n = 1, 2, \cdots$:

$$m(n + 1) > m(n) + n + 1, \tag{2}$$

$$\omega(2^{-m(n)+1}; E) < 2^{-3n^2}. \tag{3}$$

We define the closed set

$$A = R \setminus \bigcup_{n=1}^{\infty} \bigcup_{r=-\infty}^{\infty} (2^{-m(n)}r + (0, 2^{-m(n)-n-1})).$$

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The open intervals $2^{-m(n)}r+(0, 2^{-m(n)-n-1})$, $r=0, \pm 1, \pm 2, \cdots$, shall be called the *holes of order n of A*; the images of these holes under the translation $t \to t'+t$ are the *holes of order n of* $t'+A$. We now set

$$E' = [0, 1] \cap A,$$

so that E' is closed and bounded, and

$$\mu(E') \geqq 1 - \sum_{n=1}^{\infty} \sum_{r=0}^{2^{m(n)}-1} 2^{-m(n)-n-1} = 1 - \sum_{n=1}^{\infty} 2^{-n-1} = 1/2 > 0.$$

We claim that E is not essentially contained in any finite union of translates of A, let alone of E'. Assume, by contradiction, that E is essentially contained in $B=\bigcup_{j=1}^{k} (t_j+A)$ for some positive integer k and some $t_1, \cdots, t_k \in R$. From the definition of $\omega(s; F)$ and from (3) it follows in particular that

$$\omega(2^{-m(k)+1}; B) \leqq \omega(2^{-m(k)+1}; E) < 2^{-3k^2}. \tag{4}$$

Let now I be any closed interval of length $2^{-m(k)+1}$; it contains at least one complete hole of order k of t_1+A; this hole, being of length $2^{-m(k)-k-1} > 2^{-m(k+1)}$ (by condition (2)), contains at least $(2^{-m(k)-k-1}/2^{-m(k+1)})-1 \geqq 2^{m(k+1)-m(k)-k-2}$ complete holes of order $k+1$ of t_2+A; each of these holes, being of length $2^{-m(k+1)-k-2} > 2^{-m(k+2)}$ (by condition (2)), contains at least $(2^{-m(k+1)-k-2}/2^{-m(k+2)}) - 1$ $\geqq 2^{m(k+2)-m(k+1)-k-3}$ complete holes of order $k+2$ of t_3+A. Continuing in this way, we find certain holes of order $2k-1$ of t_k+A contained in holes of order $2k-2$ of $t_{k-1}+A, \cdots$, contained in a hole of order k of t_1+A; the union H of these holes of order $2k-1$ (each of length $2^{-m(2k-1)-2k}$) is therefore in $I \backslash B$. Now

$$\begin{aligned} \mu(H) &\geqq 2^{-m(2k-1)-2k} \prod_{j=1}^{k-1} 2^{m(k+j)-m(k+j-1)-k-j-1} = 2^{-m(k)-(3k^2+3k-2)/2} \\ &\geqq 2^{-m(k)-3k^2+1}. \end{aligned}$$

Since I was an arbitrary interval of length $2^{-m(k)+1}$, we have

$$\omega(2^{-m(k)+1}; B) \geqq \mu(H)/\mu(I) \geqq 2^{-3k^2},$$

and this is in contradiction to (4).

Reference

1. J. J. Schäffer, *Function spaces with translations*, Math. Ann. vol. 137 (1959) pp. 209–262.

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