DUALITY FOR BOOLEAN EQUALITIES

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Introduction. Boolean equalities occur naturally in Logic and especially in the theory of polyadic algebras. For instance, they have been used in [2] under the name of *Champ Logique*. The purpose of this paper is to develop a duality theory for Boolean equalities and by doing so establish a representation theory for them.

0. **Notation.** The basic notation for Boolean algebras is the standard one (for instance, see [1]). Throughout this paper, the letter O will denote the two-element Boolean algebra; whenever a topology is assumed to be defined on O, then this topology is the discrete one. If B is a Boolean algebra, then a *valuation* of B is a homomorphism from B onto O. We make precise the terminology we shall be using in connection with the Stone duality theory for Boolean algebras. A *Boolean space* is a compact Hausdorff space whose topology is generated by the clopen sets. If X is a Boolean space, then the *dual algebra* of X is the Boolean algebra of clopen sets of X. If B is a Boolean algebra, then the *dual space* of B is the Boolean space X of all valuations of B (the topology on X is the topology generated by all sets of the form $\{v: vp=1\}$ where p belongs to B).

1. **Definition and generalities.** A *Boolean equality* on a set Δ is a pair (B, E) where B is a Boolean algebra and E is a mapping from $\Delta\times\Delta$ into B so that, for all s, t, u in Δ,

$$E(s, s) = 1, \tag{1.1}$$

$$E(s, t) = E(t, s), \tag{1.2}$$

$$E(s, u) \wedge E(u, t) \leqq E(s, t), \tag{1.3}$$

and

(1.4) B is generated by the range of E, i.e., by $E(\Delta\times\Delta)$.

In the sequel, it will be convenient to view an *equivalence relation* on Δ as a mapping ϕ from $\Delta\times\Delta$ into O so that ϕ satisfies (1.1), (1.2) and (1.3). Clearly, if ϕ is an equivalence relation on Δ, then (O, ϕ) is a Boolean equality on Δ. Suppose (B, E) and $(\tilde{B}, \tilde{E})$ are Boolean equalities on a set Δ. A homomorphism f from B into $\tilde{B}$ shall be called an *equality homomorphism* if $fE=\tilde{E}$, i.e., if $fE(s, t)=\tilde{E}(s, t)$ for all s and t in Δ. It is easy to see that an equality homomorphism from B into $\tilde{B}$ is necessarily onto. The Boolean equalities (B, E) and $(\tilde{B}, \tilde{E})$

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are said to be *isomorphic* if there exists an equality isomorphism from B onto $\tilde{B}$. If M is an ideal in B, $\tilde{B}=B/M$, f is the natural projection from B onto $\tilde{B}$ and $\tilde{E}(s, t)=fE(s, t)$ for all s, t in Δ, then $(\tilde{B}, \tilde{E})$ is a Boolean equality on Δ; $(\tilde{B}, \tilde{E})$ shall be called a *quotient* of (B, E).

2. **Duality.** Let Δ be a nonempty set. In this section, we shall establish a one-to-one correspondence between Boolean equalities on Δ and certain sets of equivalence relations on Δ. As we shall see, this correspondence obeys all the laws of a genuine duality theory. We begin by introducing a topology on the set Φ of all equivalence relations on Δ. Note first that the set of all functions from $\Delta\times\Delta$ into O with the product topology is a Boolean space. The topology induced on Φ shall also be referred to as the *product topology*. This topology is generated by the sets of the form $\{\phi:\phi\in\Phi, \phi(s, t)=1\}$ together with their complements. We have the following basic fact.

(2.1) LEMMA. *The set Φ of all equivalence relations on Δ together with the product topology is a Boolean space.*

PROOF. It suffices to show that Φ is closed in the Boolean space of all functions from $\Delta\times\Delta$ into O. Let Φ_1, Φ_2 and Φ_3 be the sets of functions from $\Delta\times\Delta$ into O satisfying (1.1), (1.2) and (1.3) respectively. Clearly $\Phi=\Phi_1\cap\Phi_2\cap\Phi_3$. It suffices to show that Φ_1, Φ_2 and Φ_3 are closed. For instance, Φ_3 is the intersection of all sets of the form $\{\phi:\phi(s, u)=0\}\cup\{\phi:\phi(u, t)=0\}\cup\{\phi:\phi(s, t)=1\}$ as s, u and t run over Δ; since these sets are closed, it follows that Φ_3 is closed. The proof of the fact that Φ_1 and Φ_2 are closed is similar.

We are now in a position to make more precise the comment made at the beginning of this section. We shall establish a one-to-one correspondence between nonempty compact subspaces of Φ and Boolean equalities on Δ. First, we define the dual Boolean equality of a nonempty compact subspace Ψ of Φ. The *dual Boolean equality* of Ψ is the pair (B, E) where B is the dual Boolean algebra of Ψ and E is defined by

$$E(s, t) = \{\phi:\phi\in\Psi, \phi(s, t) = 1\} \quad \text{for all } s \tag{2.2}$$

and t in Δ. It is easy to check that (B, E) is indeed a Boolean equality on Δ. We now define the dual equivalence relation space of a Boolean equality (B, E). The *dual equivalence relation space* of (B, E) is a subspace Ψ of Φ so that $\phi\in\Psi$ if and only if there exists a valuation v of B for which

$$\phi(s, t) = vE(s, t) \tag{2.3}$$

for all s and t in Δ. Note that if v is a valuation of B and ϕ is defined by (2.3), then $\phi \in \Phi$ and therefore $\phi \in \Psi$. We turn now to the proof of the fact that Ψ is a Boolean space.

(2.4) THEOREM. *If Ψ is the dual equivalence relation space of a Boolean equality (B, E), then Ψ is homeomorphic with the dual space of B; in particular, this implies that Ψ is a Boolean space.*

PROOF. Let X be the dual space of B and define a one-to-one mapping χ from X onto Ψ by

$$\chi(v)(s, t) = vE(s, t) \tag{2.5}$$

for all valuations v in X and all s, t in Δ. Now observe that the topology on X is generated by the sets of the form $\{v: vE(s, t) = 1\}$ together with their complements as s and t run over Δ. The proof of the fact that χ is a homeomorphism is then completed after observing that the following identity holds:

$$\chi\{v: vE(s, t) = 1\} = \{\phi: \phi \in \Psi, \phi(s, t) = 1\}$$

for all s, t in Δ. Since X is a Boolean space, it follows that Ψ is a Boolean space.

We now show that the correspondence we have just established obeys the laws of duality.

(2.6) THEOREM. *If (B, E) is a Boolean equality on Δ and Ψ is the dual equivalence relation space of (B, E), then the dual Boolean equality of Ψ is isomorphic with (B, E); if Ψ is a Boolean subspace of Φ and (B, E) is the dual Boolean equality of Ψ, then the dual equivalence relation space of (B, E) is Ψ.*

PROOF. To prove the first part of the theorem, let $(\tilde{B}, \tilde{E})$ be the dual Boolean equality of Ψ and let X be the dual space of B. Let χ be the natural mapping from X onto Ψ defined by (2.5). If C is the dual algebra of X, f the natural isomorphism from B onto C, χ_0 the isomorphism from C onto $\tilde{B}$ induced by χ, then $\chi_0 f$ is an isomorphism from B onto $\tilde{B}$ so that $(\chi_0 f)E(s, t) = \tilde{E}(s, t)$ for all s and t in Δ. It follows that (B, E) and $(\tilde{B}, \tilde{E})$ are isomorphic. To prove the second part of the theorem, assume Ψ is a Boolean subspace of Φ and (B, E) is the dual Boolean equality of Ψ. Let X be the dual space of B and let g be the natural homeomorphism from Ψ onto X, i.e., if $\phi \in \Psi$, then $g(\phi)$ is the unique valuation of B so that, for all p in B, $g(\phi)p = 1$ if and only if ϕ belongs to the clopen set p. We have, for every ϕ in Ψ and every s, t in Δ, $g(\phi)E(s, t) = \phi(s, t)$ which shows, by (2.3), that Ψ is the dual equivalence relation space of (B, E).

(2.7) COROLLARY. *Two Boolean equalities on a set Δ are isomorphic if and only if they have the same dual equivalence relation space.*

Our next and final task in this section is to find necessary and sufficient conditions (in term of dual spaces) for a Boolean equality to be a quotient of another one.

(2.8) THEOREM. *Suppose (B, E) and $(\tilde{B}, \tilde{E})$ are Boolean equalities on Δ and suppose Ψ and $\tilde{\Psi}$ are their respective dual equivalence relation spaces. Then $(\tilde{B}, \tilde{E})$ is a quotient of (B, E) if and only if $\tilde{\Psi}$ is a subset of Ψ.*

PROOF. Suppose there exists an equality homomorphism f from B onto $\tilde{B}$. Let ϕ in $\tilde{\Psi}$ and let $\bar{v}$ be the unique valuation of $\tilde{B}$ so that $\phi(s, t) = \bar{v}\tilde{E}(s, t)$ for all s, t in Δ. Let $v = \bar{v}f$. Then v is a valuation of B and $vE(s, t) = \phi(s, t)$ which shows that $\phi \in \Psi$. Hence $\tilde{\Psi}$ is a subset of Ψ. Conversely, assume $\tilde{\Psi}$ is a subset of Ψ. In view of the preceding theorem we may and do assume that $(\tilde{B}, \tilde{E})$ and (B, E) are the dual Boolean equalities of $\tilde{\Psi}$ and Ψ respectively. Let g be the inclusion mapping of $\tilde{\Psi}$ into Ψ; since g is continuous, it induces a Boolean homomorphism $\bar{g}$ from B onto $\tilde{B}$. But

$$g^{-1}\{\phi : \phi \in \Psi, \phi(s, t) = 1\} = \{\phi : \phi \in \tilde{\Psi}, \phi(s, t) = 1\}$$

for all s and t in Δ which is the same as saying that $\bar{g}E(s, t) = \tilde{E}(s, t)$ for all s, t in Δ. Hence $\bar{g}$ is an equality homomorphism from B onto $\tilde{B}$, and that completes the proof of the theorem.

(2.9) COROLLARY. *There exists a free Boolean equality on Δ i.e., there exists a unique Boolean equality (B, E) on Δ having the property that any other Boolean equality on Δ is a quotient of (B, E).*

PROOF. The dual Boolean equality of the space Φ of all equivalence relations on Δ is the required Boolean equality.

3. Boolean equalities and transformations. In this section, as always, Δ shall be a fixed nonempty set and Φ shall be the Boolean space of all equivalence relations on Δ. A function from Δ into Δ shall be called a *transformation* (on Δ). The purpose of this section is to study the effect of transformations on Boolean equalities. We begin by extending transformations η on Δ to transformations $\bar{\eta}$ on Φ by

$$\bar{\eta}(\phi)(s, t) = \phi(\eta s, \eta t) \tag{3.1}$$

for all s, t in Δ. The mapping $\eta \to \bar{\eta}$ sends the identity onto the identity and $\eta\xi \to \bar{\xi}\bar{\eta}$ for all transformations η and ξ on Δ. Moreover,

$$(3.2)\quad \bar{\eta}^{-1}\{\phi\colon \phi\in\Phi,\ \phi(s,t)=1\}=\{\phi\colon \phi\in\Phi,\ \phi(\eta s,\eta t)=1\}$$

for all s, t in Δ. This implies that $\bar{\eta}$ is continuous on Φ for all transformations η on Δ. Suppose Ψ is a subset of Φ and G is a set of transformations on Δ. We shall say that Ψ is *invariant* under G or that G *leaves* Ψ *invariant* if $\bar{\eta}(\Psi)\subseteq\Psi$ for all η in G. This last concept allows us to formulate the main theorem of this section.

(3.3) THEOREM. *Suppose* (B, E) *is a Boolean equality on* Δ *and suppose* η *is a transformation on* Δ. *Then there exists a (necessarily unique) endomorphism* $S(\eta)$ *of* B *so that*

$$(3.4)\qquad S(\eta)E(s,t)=E(\eta s,\eta t)$$

for all s, t *in* Δ, *if and only if the dual equivalence relation space of* (B, E) *is invariant under* η.

PROOF. We may as well assume that (B, E) is the dual Boolean equality of some Boolean subspace Ψ of Φ. Assume first that Ψ is invariant under η i.e., $\bar{\eta}(\Psi)\subseteq\Psi$. Since $\bar{\eta}$ is continuous, it follows from the Stone duality theory that $\bar{\eta}$ induces an endomorphism $S(\eta)$ of B. By definition, $S(\eta)p=\bar{\eta}^{-1}p$ for every p in B (remember that the elements of B are the clopen subsets of Ψ). Therefore, for all s and t in Δ, $S(\eta)E(s,t)=E(\eta s,\eta t)$ which completes one part of the theorem. Conversely, assume there exists a (necessarily unique) endomorphism $S(\eta)$ of B so that $S(\eta)E(s,t)=E(\eta s,\eta t)$ for all s and t in Δ. Let χ be the mapping defined by (2.5). If $\phi\in\Phi$ and v is a valuation of B so that $\chi(v)=\phi$, then $\chi(vS(\eta))=\bar{\eta}\phi$ which shows that $\bar{\eta}\phi\in\Psi$ and therefore Ψ is invariant under η.

(3.5) COROLLARY. *Suppose* (B, E) *is a Boolean equality on* Δ, Ψ *the dual equivalence relation space of* (B, E) *and* G *the semi-group of transformations on* Δ *which leave* Ψ *invariant. Then there exists a unique mapping* S *from* G *into endomorphisms* $S(\eta)$ *of* B *so that* $S(\eta)E(s,t)=E(\eta s,\eta t)$ *for all* η *in* G *and all* s, t *in* Δ. *Moreover, the mapping* S *has the following two properties*: (3.6) *If* δ *is the identity on* Δ, *then* $S(\delta)$ *is the identity on* B; (3.7) *If* η *and* ξ *are in* G, *then* $S(\eta\xi)=S(\eta)S(\xi)$.

4. **Boolean equalities and operations.** If Δ is a nonempty set and n is a positive integer, then a mapping from Δ^n into Δ shall be called an *n-operation* or simply an *operation* on Δ. The purpose of this section is to establish the connection between operations on Δ and Boolean equalities on Δ.

Suppose (B, E) is a Boolean equality on Δ and T is an n-operation. Then (B, E) is said to be *compatible* with T if

$$E(s_1, t_1) \wedge \cdots \wedge E(s_n, t_n) \leqq E(T(s_1, \cdots, s_n), T(t_1, \cdots, t_n))$$

for all $s_1, \cdots, s_n, t_1, \cdots, t_n$ in Δ; E itself is said to be compatible with T if (B, E) is compatible with T.

It is easy to see that if T is an operation on Δ, then the space of equivalence relations compatible with T is compact. This, combined with the following theorem, implies that there exists a free Boolean equality on Δ which is compatible with T (i.e., free with respect to the property of being compatible with T).

(4.1) THEOREM. *Suppose (B, E) is a Boolean equality on a set Δ and suppose Ψ is the dual equivalence relation space of (B, E). If T is an operation on Δ, then (B, E) is compatible with T if and only if every equivalence relation ϕ in Ψ is compatible with T.*

PROOF. Using (2.2) and (2.3), the proof reduces to a straightforward verification.

We conclude by making a few remarks concerning Boolean equalities on familiar algebraic structures. Suppose, for instance, (B, E) is a Boolean equality on a ring Δ and assume moreover that (B, E) is compatible with the ring operations. By (4.1), if ϕ is in the dual of (B, E), then ϕ corresponds to a uniquely determined ideal in Δ. In other words, to every Boolean equality on Δ compatible with the ring operations there corresponds a uniquely determined set of ideals in Δ. Such sets of ideals are easily characterized.

REFERENCES

1. P. R. Halmos, *The basic concepts of algebraic logic*, Amer. Math. Monthly vol. 63 (1956) pp. 363–387.

2. J. Łos, *Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres*, Mathematical interpretation of formal systems, Amsterdam, North-Holland Publishing Co., 1955, pp. 98–113.

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