ON A CLASS OF QUADRATIC ALGEBRAS

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In a recent paper [3] concerning cubic forms which permit a new type of composition, a class of nonassociative algebras A with 1 over F of characteristic $\neq 2$ is encountered: central simple flexible quadratic algebras satisfying the identity

$$[x, [x, y]] = 0 \qquad \text{for all } x, y \text{ in } A. \tag{1}$$

We shall denote this class by $\mathfrak{C}$. The commutative algebras in $\mathfrak{C}$ are the well-known central simple Jordan algebras of degree two. In [3] an example is given (over F containing $\sqrt{-1}$ of a 7-dimensional algebra in $\mathfrak{C}$ which is not commutative. We shall denote by $\mathfrak{C}'$ the class of all algebras in $\mathfrak{C}$ which are not commutative.

In this note we show that, in the presence of the other conditions on the algebras in $\mathfrak{C}$, identity (1) may be replaced by

$$[x, y]^2 = 0 \qquad \text{for all } x, y \text{ in } A. \tag{2}$$

For arbitrary F of characteristic $\neq 2$ and for any dimension $\geqq 7$ (possibly infinite), we show that $\mathfrak{C}'$ contains an algebra A of dimension n over F. On the other hand, if F is also of characteristic $\neq 3$, then $\mathfrak{C}'$ contains no algebra of dimension $\leqq 6$.

An algebra A (of possibly infinite dimension) with 1 over a field F is called a *quadratic algebra* in case $A \neq F1$ and, for every x in A, there are $t(x)$, $n(x)$ in F such that

$$x^2 - t(x)x + n(x)1 = 0. \tag{3}$$

It is well-known that, by defining $t(\alpha 1) = 2\alpha$, $n(\alpha 1) = \alpha^2$, the *trace* $t(x)$ is linear and the *norm* $n(x)$ is a quadratic form. If F has characteristic $\neq 2$, there is a (possibly infinite) basis $U = \{u_i\}$ of A over F such that $1 = u_0 \in U$,

$$u_i^2 = \beta_i 1, \qquad \beta_i \in F, \beta_0 = 1, \tag{4}$$

and, for $i \neq 0, j \neq 0, i \neq j$,

$$u_i u_j = \sum \pi_{ijk} u_k = -u_j u_i \qquad (i \neq j; i \neq 0, j \neq 0) \tag{5}$$

where only a finite number of the π_{ijk} are $\neq 0$. From (5) we have

$$\pi_{ijk} = -\pi_{jik} \text{ for all } k \qquad (i \neq j; i \neq 0, j \neq 0). \tag{6}$$

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Any $x \in A$ may be written uniquely in the form

$$x = \sum \alpha_i u_i \tag{7}$$

with only a finite number of the $\alpha_i \neq 0$. Then

$$t(x) = 2\alpha_0, \qquad n(x) = \alpha_0^2 - \sum_{i \neq 0} \alpha_i^2 \beta_i. \tag{8}$$

It is well-known that $n(x)$ is nondegenerate if and only if in (4) we have $\beta_i \neq 0$ for every i. One sees easily [2; 3] that A is central simple if and only if $n(x)$ is nondegenerate and the dimension of A is $\geqq 3$. Clearly A is commutative if and only if all $\pi_{ijk} = 0$ in (5).

An algebra A over F is called *flexible* in case

$$(xy)x = x(yx) \qquad \text{for all } x, y \text{ in } A. \tag{9}$$

It is shown in [1, p. 588] that a quadratic algebra is flexible if and only if

$$\pi_{ij0} = \pi_{iji} = \pi_{ijj} = 0 \qquad (i \neq j;\ i \neq 0, j \neq 0) \tag{10}$$

and

$$\beta_i \pi_{jki} = \beta_k \pi_{ijk} \qquad (i, j, k \text{ distinct};\ i \neq 0, j \neq 0, k \neq 0) \tag{11}$$

are satisfied in (5). Hence the algebras in $\mathfrak{C}$ are the algebras A of dimension $\geqq 3$ over F of characteristic $\neq 2$ satisfying (4) with all $\beta_i \neq 0$, (5), (6), (10), (11), and (1) where $[x, y]$ is the *commutator* $[x, y] = xy - yx$. The algebras in $\mathfrak{C}'$ satisfy the additional requirement that at least one π_{ijk} in (5) is $\neq 0$.

Let (x, y) be the nondegenerate symmetric bilinear form such that $n(x) = (x, x)$. In [3, equation (78)] it is shown that, for any A in $\mathfrak{C}$, we have

$$n(xy) = (xy, yx) = n(yx) \qquad \text{for all } x, y \text{ in } A.$$

Hence $n([x, y]) = (xy - yx, xy - yx) = n(xy) - 2(xy, yx) + n(yx) = 0$ for all x, y in A. Since also $t([x, y]) = 0$ [3, p. 172], we see that (3) implies (2). But then we may replace (1) by (2) in the definition of $\mathfrak{C}$. For, conversely, we may linearize (2) in y to obtain

$$[x, y][x, z] + [x, z][x, y] = 0. \tag{12}$$

In [3, p. 172] it is shown that flexibility implies

$$t(xy) = t(yx), \qquad t((xy)z) = t(x(yz)),$$

and

$$(x, y) = \frac{1}{2} t(x \bar{y}) \quad \text{where} \quad \bar{y} = t(y)1 - y.$$

Hence (12) implies

$$\begin{aligned} 0 &= t([x, y][x, z] + [x, z][x, y]) \\ &= 2t([x, y][x, z]) \\ &= 2t((xy)(xz) - (zx)(xy) - (yx)(xz) + (zx)(yx)) \\ &= 2t((xyx)z - z(x(xy)) - ((yx)x)z + z(xyx)) \\ &= (2xyx - x(xy) - (yx)x, \bar{z}) \end{aligned}$$

for every z in A. Since (x, y) is nondegenerate, we have $2xyx - x(xy) - (yx)x = 0$, implying (1).

Although (1) has the advantage of being an identity of lower degree than identity (2), the latter may be very easily stated: the square of every commutator is 0. We shall use this in the proof of the Theorem below.

LEMMA. *Let A_1 over F be an algebra in $\mathfrak{C}'$, and let the dimension of A_1 be m. Then, for any $n \geqq m$ (possibly infinite), there exists an n-dimensional algebra A over F in $\mathfrak{C}'$ such that A_1 is a subalgebra of A.*

PROOF. A basis U_1 for A_1 (of the special form we have chosen) may be extended to a basis $U = U_1 \cup U_2$ for an algebra A over F such that A has dimension n over F. It remains to define multiplication appropriately for pairs of elements in U where at least one of the elements of the pair is in U_2. Define

$$u_i^2 = -1 \qquad \text{for } u_i \text{ in } U_2, \tag{13}$$

and

$$u_i u_j = 0 \qquad \text{for } i \neq j, \text{ if either } u_i \text{ or } u_j \text{ is in } U_2. \tag{14}$$

All of the conditions for A to be in $\mathfrak{C}'$ are obviously satisfied, except possibly for (11) and (2) which we verify as follows. If u_i, u_j, and u_k are all in U_1, then (11) is satisfied since A_1 is in $\mathfrak{C}$. If both u_i and u_j are in U_1, then $u_i u_j$ in A_1 implies $\pi_{ijk} = 0$ for all k for which $u_k \in U_2$; hence (14) implies (11) in this case. Finally, if either u_i or u_j is in U_2, then $\pi_{ijk} = \pi_{jki} = 0$ by (14). To verify (2), we write x in A in the form $x = x_1 + x_2$ where $x_1 \in A_1$ and where x_2 is a linear combination of elements of U_2. With $y = y_1 + y_2$ written similarly, we obtain $[x, y] = [x_1, y_1]$ by (14). Then $[x, y]^2 = [x_1, y_1]^2 = 0$ since A_1 is in $\mathfrak{C}$.

THEOREM. *Let F be an arbitrary field of characteristic $\neq 2$. For any dimension $n \geqq 7$ (possibly infinite), there exists an n-dimensional algebra*

A over F in the class $\mathfrak{C}'$. If F is also of characteristic $\neq 3$, there are no algebras A of dimension $\leqq 6$ over F in $\mathfrak{C}'$.

PROOF. We use the lemma to see that (i) the first statement in the conclusion of the Theorem may be established by constructing a 7-dimensional algebra in $\mathfrak{C}'$, while (ii) the final statement may be proved by showing that any 6-dimensional algebra in $\mathfrak{C}$ is commutative (assuming characteristic $\neq 3$).

To construct an example in (i), let $1, u_1, u_2, \cdots, u_6$ be a basis for A over F. Define

$$u_i^2 = -1, \quad u_{i+3}^2 = 1, \quad u_iu_{i+3} = u_{i+3}u_i = 0 \qquad (i = 1, 2, 3).$$

For cyclic permutations i, j, k of 1, 2, 3, define

$$\begin{aligned} u_iu_j &= u_iu_{j+3} = u_{i+3}u_j = u_{i+3}u_{j+3} = u_k - u_{k+3} = -u_ju_i \\ &= -u_{j+3}u_i = -u_ju_{i+3} = -u_{j+3}u_{i+3}. \end{aligned}$$

Then (4) with all $\beta_i \neq 0$, (5), (6), (10) and (11) are satisfied. In order to verify (1) we note that, relative to the basis $1, u_1, u_2, \cdots, u_6$, the matrix T_x of the linear transformation $y \rightarrow [x, y]/2$ has the form

$$T_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & K & -K \\ 0 & K & -K \end{pmatrix},$$

where

$$K = \begin{pmatrix} 0 & -\alpha_3 - \alpha_6 & \alpha_2 + \alpha_5 \\ \alpha_3 + \alpha_6 & 0 & -\alpha_1 - \alpha_4 \\ -\alpha_2 - \alpha_5 & \alpha_1 + \alpha_4 & 0 \end{pmatrix} \qquad \text{for } x \text{ in (7).}$$

Hence $T_x^2 = 0$, implying (1).

To prove (ii), we may as well assume that F is algebraically closed. For, if Σ is the algebraic closure of F, and if A over F is in the class $\mathfrak{C}'$, then A_Σ is in the corresponding class of algebras over Σ (the equations (4), (5), (6), (10), and (11) remain unchanged, and one may linearize (1) to obtain an equivalent multilinear identity). Hence we may assume that $\beta_i = -1$ $(i = 1, \cdots, 5)$ in (4). The 60 elements π_{ijk} which are not given as 0 by (10) are partitioned by the relations (6) and (11) into 10 classes, within each of which the elements differ only by a factor of ± 1. Writing subscripts modulo 5 (using 5, not 0), we make the following abbreviations for representatives of these 10 classes:

$$\sigma_i = \pi_{i+1,i+2,i+3}, \qquad \epsilon_i = \pi_{i+1,i+3,i+4} \qquad (i = 1, \cdots, 5). \tag{15}$$

Then, for subscripts modulo 5, we have

$$u_iu_{i+1} = \sigma_{i+4}u_{i+2} + \epsilon_{i+2}u_{i+3} + \sigma_{i+3}u_{i+4} \qquad (i = 1, \cdots, 5) \tag{16}$$

and

$$u_iu_{i+3} = -\epsilon_{i+2}u_{i+1} - \epsilon_{i+4}u_{i+2} + \sigma_{i+2}u_{i+4} \qquad (i = 1, \cdots, 5). \tag{17}$$

Now (2), (5) and (4) imply $[u_i, u_j]^2 = (2u_iu_j)^2 = -4\Sigma\pi_{ijk}^2 = 0$ for $i \neq j$. Hence (16) implies

$$\sigma_{i+4}^2 + \epsilon_{i+2}^2 + \sigma_{i+3}^2 = 0 \qquad (i = 1, \cdots, 5), \tag{18}$$

and (17) implies

$$\epsilon_{i+2}^2 + \epsilon_{i+4}^2 + \sigma_{i+2}^2 = 0 \qquad (i = 1, \cdots, 5). \tag{19}$$

Eliminating the ϵ's in (18) and (19), and adding the resulting equations, we arrive easily at $\sigma_i = \epsilon_i = 0$ $(i = 1, \cdots, 5)$ since the characteristic of F is $\neq 2, 3$. Hence A is commutative.

REMARK. Characteristic $\neq 3$ is not required in a direct proof that there are no algebras in $\mathfrak{C}'$ of dimension $\leqq 5$.

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