$r \geqq \sum_p r(p)$ then G is quotient divisible.

Proof. For each p, G/pG is a direct sum of cyclic groups of order p: $G/pG = \sum_{(i,p)\in I(p)} Z(x(i, p)+pG)$ for some elements $x(i, p)\in G$. Evidently, $r(p) = |I(p)|$. Since rank pG = rank $G = r$ for each p, we may apply Theorem 2.1 with $N = \{p \mid p \text{ is a prime}\}$, $A_p = pG$ for each $p\in N$ and $B_p = G$ for each $p \in N$, and obtain a free group $F \subseteq G$ such that $F+pG = G$ for each p. Applying Lemma 3.3, with S any basis of F, we conclude that G is quotient divisible.

Corollary 3.5. *Any torsion free group of infinite rank is quotient divisible.*

Proof. $\sum_p r(p) \leqq \sum_p r = \aleph_0 r = r$.

Bibliography

1. R. A. Beaumont and R. S. Pierce, *Torsion free rings*, Illinois J. Math. **5** (1961), 61–98.

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CONVOLUTIONS OF SLOWLY OSCILLATING FUNCTIONS

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1. **Introduction.** In the study of asymptotic formulae for arithmetic functions we invariably come upon such functions as $x^\alpha \log^\beta x$, li x, $x^\alpha(\log x)^\beta(\log\log x)^\gamma$, $x^\alpha \exp(\beta \log^{\gamma_0} x)$. If α is complex and β and γ are real numbers ($\gamma_0<1$) then these functions are of the form $x^\alpha L(x)$, where L is a *slowly oscillating* function; i.e., a continuous positive valued function on $[x_0, \infty)$ for some $x_0 \geqq 1$ such that

$$\lim_{x\to\infty} L(cx)/L(x) = 1 \tag{1}$$

for each $c>0$. A common approach to asymptotic formulae is the convolution method of Landau which was formulated into a general theorem by the author [**4**]. The resulting main terms involve convolutions of functions $x^\alpha L(x)$. In the present paper we shall give conditions under which such convolutions are also in the form $x^\alpha L(x)$ or nearly so.

Now it is known [**1**; **3**] (see [**2**] for other properties) that a function L on $[x_0, \infty)$ is slowly oscillating if and only if there exist continuous functions ρ and δ on $[x_0, \infty)$ such that $\rho(x)>0$, $\rho(x)\to\rho_0>0$, $\delta(x)\to 0$ as $x\to\infty$ and

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$$L(x) = \rho(x) \exp \int_{x_0}^{x} t^{-1}\delta(t)dt. \tag{2}$$

We shall be concerned in particular with the *special* slowly oscillating functions having the form

$$L(x) = \rho \exp \int_{1}^{x} t^{-1}\delta(t)dt, \tag{3}$$

where $\rho > 0$ is constant and δ is a bounded measurable function with $\delta(x) = o(1)$ as $x \to \infty$. By a *convolution* of two functions A and B we shall mean one of

$$\int_{a}^{x/a} A(x/u)dB(u) \quad \text{or} \quad \int_{a}^{x/a} k(u)A(x/u)B(u)du. \tag{4}$$

The former of these expressions is called the *Stieltjes resultant* of A by B on $[a^2, \infty)$.

2. **A convolution theorem.** *Suppose α and β are real numbers with $\beta \neq 0$ and that L and M are slowly oscillating with M special. Then for sufficiently large $a \geqq 1$ the Stieltjes resultant on $[a^2, \infty)$ of $x^\alpha L(x)$ by $x^\beta M(x)$ has the form $\beta x^\gamma N(x)$ where $\gamma = \max(\alpha, \beta)$ and $N(x)$ is slowly oscillating for $x > a^2$. Further, $N(x)$ is asymptotically proportional to $L(x)$ if $\alpha > \beta$ and to $M(x)$ if $\alpha < \beta$.*

Proof. If M is special then we have

$$M(x) = \rho \exp \int_{1}^{x} t^{-1}\delta(t)dt$$

with $\rho > 0$, $\delta(t) = o(1)$ as $t \to \infty$. Choose $a \geqq 1$ so that $|\delta(t)| < |\beta|$ for all $t \geqq a$ and thus $\beta + \delta(t)$ remains constant in sign for $t \geqq a$. Now the Stieltjes resultant in question is

$$\int_{a}^{x/a} (x/u)^\alpha L(x/u)d(u^\beta M(u))$$

$$= \int_{a}^{x/a} (x/u)^\alpha L(x/u)u^{\beta-1}M(u)(\beta + \delta(u))du$$

$$= x^\alpha \int_{a}^{x/a} u^{-(\alpha-\beta)-1}L(x/u)M(u)(\beta + \delta(u))du \tag{1}$$

$$= x^\beta \int_{a}^{x/a} u^{-(\beta-\alpha)-1}M(x/u)(\beta + \delta(x/u))L(u)du. \tag{2}$$

If $\alpha>\beta$ we apply Lemma 1 below to (1) with

$$A(x) = x^{-(\alpha-\beta)-1}M(x)(\beta + \delta(x)) = O(x^{-(\alpha-\beta)-1+\epsilon})$$

for each $\epsilon>0$ (e.g., $\epsilon=(\alpha-\beta)/2$). Thus the resultant is asymptotic to

$$x^\alpha L(x)\int_a^\infty u^{-(\alpha-\beta)-1}M(u)(\beta + \delta(u))du$$

as $x\to\infty$, if $\alpha>\beta$. (The integral is different from 0 by our choice of a.) Similarly, if $\beta>\alpha$, we apply Lemma 1 to (2) and find the resultant asymptotic to

$$\beta x^\beta M(x)\int_a^\infty u^{-(\beta-\alpha)-1}L(u)du.$$

If $\alpha=\beta$, (1) gives

$$x^\alpha\int_a^{x/a} u^{-1}L(x/u)M(u)(\beta + \delta(u))du,$$

to which we apply Lemma 2.

3. **Two lemmas.** LEMMA 1. *Suppose A and B are measurable functions on $[a, \infty)$ $(a>0)$ with*

$$A(x) = O(x^{-\kappa})$$

for some $\kappa>1$. Suppose B is positive valued, bounded on each bounded interval and asymptotic to a slowly oscillating function. Then

$$\lim_{x\to\infty}\int_a^{x/a} A(u)(B(x/u)/B(x))du = \int_a^\infty A(u)du. \tag{1}$$

PROOF. Set $B(x)=0$ for $0\leqq x<a$ and apply the Lebesgue dominated convergence theorem to

$$\int_a^{x/a} A(u)(B(x/u)/B(x))du = \int_a^\infty A(u)(B(x/u)/B(x))du.$$

Now for all $u\geqq a$

$$\lim_{x\to\infty} A(u)B(x/u)/B(x) = A(u). \tag{2}$$

With the aid of (1.2), for each $\epsilon>0$

$$B(x/u)/B(x) = O(u^\epsilon)$$

uniformly for $u\geqq a$, $x\geqq K$ (sufficiently large), and so we have

$$A(u)B(x/u)/B(x) = O(u^{-(\kappa-\epsilon)}) \tag{3}$$

with $0<\epsilon<\kappa-1$. Thus since $u^{-(\kappa-\epsilon)}$ is absolutely integrable over $[a, \infty)$, (2) and (3) imply (1).

LEMMA 2. *Suppose L is slowly oscillating and B is measurable and asymptotic to a slowly oscillating function. Then for sufficiently large a*

$$N(x) = \int_a^{x/a} u^{-1}L(x/u)B(u)du = \int_a^{x/a} u^{-1}L(u)B(x/u)du$$

is slowly oscillating for $x \geqq x_0 > a^2$.

The proof is elementary. We use the fact that $\int_a^x u^{-1}L(u)du$ is slowly oscillating and that

$$\lim_{x\to\infty} B(cx)/B(x) = 1$$

uniformly for $k_1 \leqq c \leqq k_2$ for each $k_2 > k_1 > 0$.

4. Remarks. If $\alpha=\beta=0$ we have the Stieltjes resultant of L by M. We can show it slowly oscillating under various hypotheses. In particular if L and M are both special and nondecreasing (M nonconstant).

Note that A in Lemma 1 need not be real-valued. Thus if α and β are complex with $\Re\alpha \neq \Re\beta$ the Stieltjes resultant of $x^\alpha L(x)$ by $x^\beta M(x)$ is $x^\gamma N(x)$ with $N(x)$ complex-valued but asymptotically proportional to $L(x)$ or $M(x)$ according as $\Re\alpha > \Re\beta$ or $\Re\beta > \Re\alpha$. γ would be α in the former case and β in the latter.

BIBLIOGRAPHY

1. J. Karamata, *Sur une mode de croissance regulière des fonctions*, Mathematica (Cluj) **4** (1930), 38–53.

2. E. E. Kohlbecker, *Weak asymptotic properties of partitions*, Trans. Amer. Math. Soc. **88** (1958), 346–365.

3. J. Korevaar, T. van Aardenne-Ehrenfest, and N. G. de Bruijn, *A note on slowly oscillating functions*, Nieuw Arch. Wiskunde **23** (1949), 77–86.

4. J. P. Tull, *Dirichlet multiplication in lattice-point problems*. II. Pacific J. Math. **9**, no. 2 (1959), 609–615.

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