# LINEARLY ORDERABLE SPACES[1]

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A *linearly ordered* space $(X, T, <)$ is a set $X$ on which a linear ordering $<$ has been defined such that the interval topology generated by $<$ agrees with $T$. (A subbasic set for the interval topology induced by $<$ is everything less than or greater than a given point.)

A space $(X, T)$ is *linearly orderable* if there is a linear ordering $<$ on $X$ whose associated interval topology agrees with $T$.

The subspace $\{-1\} \cup \{n^{-1} : n = 1, 2, \cdots \}$ of the real line $R$ is not linearly ordered with respect to the usual ordering of $R$ because $\{-1\}$ is open in the relative topology but not in the interval topology. However, this subspace is homeomorphic to the linearly ordered space $\{n^{-1} : n = 1, 2, \cdots \}$, so it is linearly orderable.

The subspace $(0, 1) \cup \{2\}$ of $R$ is, however, not even linearly orderable.

To the best of my knowledge there are no known sufficient conditions for a disconnected space to be linearly orderable. In the connected case the only affirmative results are the well-known characterizations of the arc (cf., e.g. [2, p. 168]) and a result of Eilenberg [1]. The purpose of this note is to prove the following:

THEOREM. *Every zero-dimensional, separable metrizable space is linearly orderable.*

PROOF. Without loss of generality we may assume our space $(X, T)$ is a subset of the Cantor set $C$ in the unit interval $I$ [3, p. 173]. We will construct a homeomorphism $f$ of $X$ into $C$ which maps a countable dense subset of $X$ into a carefully selected subset of those points of $C$ which are end points of intervals of $R \sim C$. We will then show that in $f[X]$:

(a) each point is the $T$-limit of a monotone increasing sequence of points of $f[X]$, or has an immediate predecessor, or is the minimum of $f[X]$, and

(b) each point is the $T$-limit of a monotone decreasing sequence of points of $f[X]$, or has an immediate successor, or is the maximum of $f[X]$.

The $T$ and interval topologies coincide precisely when (a) and (b) hold.

Select a point $a_0$ in $X$.

Let $I_0$ be that closed end third of $I$ containing $a_0$ and $I_2$ be the other. If $XI_2 \neq \varnothing$, select a point $a_1$ in $XI_2$.

Let $I_{00}$ be that closed end third of $I_0$ containing $a_0$ and $I_{02}$ be the other. If $XI_{02} \neq \varnothing$, select a point $a_{1/3}$ in $XI_{02}$. Similarly let $I_{22}$ be that closed end third of $I_2$ containing $a_1$ and $I_{20}$ be the other. If $XI_{20} \neq \varnothing$, select a point $a_{2/3}$ in $XI_{20}$.

Proceeding inductively, for each integer $n > 1$, define intervals $I_{p_1 \cdots p_n}$ ($p_i = 0$ or $2$ for $i = 1, 2, \cdots, n$) and, if $XI_{p_1 \cdots p_n} \neq \varnothing$, a selected point in $XI_{p_1 \cdots p_n}$ subject to the following restrictions:

$I_{p_1 \cdots p_n}$ is a closed end third of $I_{p_1 \cdots p_{n-1}}$.

The selected point in $XI_{p_1 \cdots p_n}$ is the selected point in $XI_{p_1 \cdots p_{n-1}}$ if $p_n = p_{n-1}$.

The selected point in $XI_{p_1 \cdots p_n}$ has index $u \cdot 3^{-n+1} + \sum_{i=1}^{n-1} p_i \cdot 3^{-i}$, where $u = 0$ if $p_n = 0$, and $u = 1$ otherwise.

Note that for each $a \in X$, there is a unique sequence of intervals $\{I_{p_1 \cdots p_n}\}$ such that $\{a\} = \bigcap_{n=1}^{\infty} I_{p_1 \cdots p_n}$. Thus, we may define a mapping $f: X \to C$ by letting $f(a) = \sum_{i=1}^{\infty} p_i \cdot 3^{-i}$.

If $a$ and $b$ are distinct points of $X$, there is an integer $j$ such that $p_j \neq q_j$. Thus, $\sum_{i=1}^{\infty} p_i \cdot 3^{-i} \neq \sum_{i=1}^{\infty} q_i \cdot 3^{-i}$. So, $f$ is a one-one mapping.

Moreover, $f$ is continuous. For, if $\{b_m\}$ is a sequence of points of $X$ converging to $a \in \bigcap_{n=1}^{\infty} I_{p_1 \cdots p_n}$, then there is an integer $N_n$ such that both $b_m$ and $a$ are in $I_{p_1 \cdots p_n}$ if $m \geq N_n$, whence both $f(b_m)$ and $f(a)$ are in $[\sum_{i=1}^{n} p_i \cdot 3^{-i}, 3^{-n} + \sum_{i=1}^{n} p_i \cdot 3^{-i}]$. A similar argument shows that $f^{-1}$ is also continuous.

We will show next that if a point of $f[X]$ which is an end point of an interval of $R \sim C$ is not a $T$-limit point from above (below), then it is not a limit point from above (below) in the interval topology. The converse is clear. It is easily seen from the nature of the mapping $f$ that each point of $f[X]$ which is not an end point of an interval of $R \sim C$ is a two-sided $T$-limit point of $f[X]$. So, this will conclude the proof.

Let $x = \sum_{i=1}^{r} p_i \cdot 3^{-i} \in f[X]$, where $p_r \neq 0$. Thus $p_r = 1$ or $2$, and $p_i = 0$ or $2$ for $i \neq r$. Clearly $x$ is a $T$-limit point from at most one side. Thus, by symmetry, it suffices to prove our assertion in case $p_r = 2$ and $x$ is not a $T$-limit point from above. The details of the following argument may be filled in easily.

Let $y = \inf \{z \in f[X] : x < z\}$. Suppose first that there is a largest integer $n_0 > r$ such that $\alpha = x + 3^{-n_0} \in f[X]$. If there is also a smallest $s_0$ such that $\beta = x + \sum_{i=n_0+1}^{s_0} 2 \cdot 3^{-i} \in f[X]$, then $y = \beta$, while if there is no such $s_0$, then $y = \alpha$. If there is no such $n_0$ and there is a smallest $s_0$ such that $\gamma = x + \sum_{i=r+1}^{s_0} 2 \cdot 3^{-i} \in f[X]$, then $y = \gamma$, while if there is

no such $s_0$ then $y = 3^{-r+1} + \sum_{i=1}^{r-1} p_i \cdot 3^{-i} \in f[X]$. Thus $y$ is the immediate successor of $x$.

Let $w = \sup\{z \in f[X] : z < x\}$, let $u_j = \sum_{i=1}^{r-1} p_i \cdot 3^{-i}$ for $j = 1, 2, \cdots,$ $r-1$, let $u_r = 0$, and $u_k$ be that $u_j$ ($j = 1, 2, \cdots, r$) with least index in $f[X]$. If there is a least $n_0 \geq r - k + 1$ such that $\delta = u_k + 3^{-n_0} \in f[X]$, then $w = \delta$, while if there is no such $n_0$, then $w = u_k$. Thus $w$ is the immediate predecessor of $x$.

A similar procedure is followed in case $x = 0$ or $1$.

In a future note the case of arbitrary subspaces of $R$ will be discussed. (See [4].)

<div align="center">REFERENCES</div>

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