EMBEDDING NUMBERS FOR FINITE GROUPS

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This note is concerned with the following problem. Let H denote a subgroup of a finite group G and let L denote a linear or one dimensional representation (i.e., a character) of H. We assume throughout that the field F is algebraically closed and is either of characteristic 0 or of prime characteristic which does not divide the order of any groups under consideration. Let $G|L$ denote the corresponding induced representation of G. How many distinct (i.e., nonequivalent) irreducible representations appear in the decomposition of $G|L$ into irreducible parts? (This number is just the central intertwining number of $G|L$, which is denoted by $\mathcal{C}\mathcal{I}(G|L)$. Cf. [1].) More specifically, we are interested in determining an upper bound on the number of distinct irreducible representations which will appear, purely in terms of the way H is embedded in G, and in terms which do not depend on the particular linear representation L of H. Two such bounds come quickly to mind. The number of classes (of conjugates) of the super group G, which we denote $\{G: e\}$, is clearly an upper bound. Dimension considerations also give $[G: H]$ as an upper bound. We now introduce a new group theoretic invariant which heuristically is a measure of the manner in which the classes of G are distributed among the H-cosets of G.

DEFINITION. Let H be a (not necessarily normal) subgroup of a finite group G. For each normal subset N of G, let $\phi_1(N)$ denote the number of classes (of conjugates) of G contained in N. Let $\phi_2(N)$ denote the number of right H-cosets of G which have nonzero intersection with N. Let $\phi(N) = \{G: e\} - \phi_1(N) + \phi_2(N)$. We then define the ***embedding number*** of H in G, denoted by $(G: H)$, to be the minimum of the $\phi(N)$, as N is taken over all normal subsets of G. We remark that a definition of ϕ_2 using left cosets would yield the same value for $(G: H)$ since N^{-1} intersects the same number of left cosets as N does right cosets.

Taking $N = \{e\}$ where e is the identity element of the group we have $(G: H) \leqq \{G: e\}$. Taking $N = G$ we have $(G: H) \leqq [G: H]$. If $H \neq G$, it is easy to verify that $(G: H) > 1$. If H is a proper normal subgroup, then, taking $N = H$ we have $(G: H) < \{G: e\}$. In the case where H is a normal subgroup of G, another number associated with the embedding of H in G is the number of classes in the factor group G/H. We call this the *class number* of H in G and denote it by $\{G: H\}$.

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PROPOSITION 1. *If H is a normal subgroup of a finite group G, then* $\{G:H\} \leqq (G:H)$.

PROOF. Suppose $(G:H)=m$. Then there exists a normal subset, say N of G, such that N contains n classes of G and intersects c H-cosets such that $m=\{G:e\}-n+c$. Note that $n \geqq c$ since $m=(G:H) \leqq \{G:e\}$. Let $\overline{N}$ denote the smallest normal subset of the factor group G/H, containing the c H-cosets which have nonzero intersection with N. Then the number of classes of G/H contained in $\overline{N}$ is less than or equal to c. Let $N'=\{x: x \in G$ and x is contained in some H-coset belonging to $\overline{N}\}$. Then N' is a normal subset of G such that $N \subset N'$ and thus N' contains at least n classes of G. Then $(G-N')$ is a normal subset of G and contains at most $\{G:e\}-n$ classes of G. Thus $(G/H-\overline{N})$ contains at most $\{G:e\}-n$ classes of G/H. Further $\overline{N}$ contains at most c classes. Hence G/H contains at most $\{G:e\}-n+c$ $=m$ classes. Hence $\{G:H\} \leqq (G:H)$.

Thus in general we have $\{G:H\} \leqq (G:H) \leqq [G:H]$. If G/H is abelian this degenerates to $\{G:H\}=(G:H)=[G:H]$. Now that we have a relative idea of how this new "embedding number" compares with the group theoretic invariants usually associated with the embedding of H in G, we proceed to show the significance of $(G:H)$ in the theory of monomial representations. We must first prove a preliminary result.

LEMMA. *Let H be a subgroup of a finite group G and let L denote a linear representation of H, over the field F. Let $G|L$ denote the corresponding induced representation of G. Let $D_1, D_2, \cdots, D_{n+1}$ denote distinct classes of G and let $S_i=\sum_{x\in D_i}(G|L)_x$, for $i=1,2,\cdots,n+1$. If these $n+1$ classes are completely contained in the union of n right H-cosets of G, then the S_i, $i=1,2,\cdots,n+1$, are linearly dependent over F.*

PROOF. Index the right H-cosets of G, $\{H\sigma_j\}$, $j=1,\cdots,k$, in such a way that $D_i \subset \bigcup_{j=1}^{n} H\sigma_j$, for $i=1,2,\cdots,n+1$. Then $\{\sigma_j^{-1}: j=1,\cdots,k\}$ form a set of representatives of the left H-cosets of G. By [1, Corollary to Theorem 3], it is sufficient to show that there exists $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$ in F, not all zero, such that $\sum_{i=1}^{n+1}\alpha_i\beta_{ij}$ $=0$ for $j=1,\cdots,k$, where $\beta_{ij}=\sum_{x\in\sigma_j^{-1}D_i\cap H} L_x$ and $\beta_{ij}=0$ if $\sigma_j^{-1}D_i$ $\cap H$ is empty. Since L is linear we have $\beta_{ij}\in F$. Consider the set of homogeneous linear equations

$$\sum_{i=1}^{n+1} \beta_{ij}x_i = 0, \qquad j = 1, \cdots, n.$$

This system has n equations and $n+1$ unknowns and thus has a nontrivial solution, say $x_i=\alpha_i\in F$. Hence $\sum_{i=1}^{n+1}\alpha_i\beta_{ij}=0$ for $j=1,\cdots,n$. By our indexing of the H-cosets we have that $\sigma_j^{-1}D_i\cap H=D_i\sigma_j^{-1}\cap H=(D_i\cap H\sigma_j)\sigma_j^{-1}$ is empty (and thus $\beta_{ij}=0$), for $j>n$ and $i=1,\cdots,n+1$. Hence $\sum_{i=1}^{n+1}\alpha_i\beta_{ij}=0$ for $j=1,\cdots,k$.

THEOREM. *Let H denote a subgroup of a finite group G and let L be a linear representation of H. Then the number of distinct irreducible representations appearing in the decomposition of the induced representation $G|L$ is less than or equal to $(G:H)$.*

PROOF. There exists a normal subset N of G such that $(G:H)=n-m+\phi_2(N)$, where $n=\{G:e\}$ and $m=\phi_1(N)$. Let $C_1, C_2,\cdots,C_m$ denote the classes of G which are contained in N and let $C_{m+1}, C_{m+2},\cdots,C_n$ denote the remaining classes of G. Let $S_i=\sum_{x\in C_i}(G|L)_x$, for $i=1,2,\cdots,n$. By the previous lemma there are at most $\phi_2(N)$ elements among the S_i, $i\leqq i\leqq m$, which are linearly independent over the field F. Hence there are at most $n-m+\phi_2(N)=(G:H)$ linearly independent elements among the S_i, $1\leqq i\leqq n$. By [1, Theorem 1], $\mathcal{CI}(G|L)\leqq(G:H)$. That is to say, the number of distinct irreducible representations appearing in the decomposition of $G|L$ is less than or equal to $(G:H)$.

COROLLARY. *Let H denote an abelian subgroup of a finite group G. Then $\{G:e\}\leqq(G:H)[H:e]$.*

PROOF. Let L denote the regular representation of H. Then $\mathcal{CI}(L)=\{H:e\}=[H:e]$ and each irreducible representation appearing in the decomposition of L is linear. Thus by the theorem $\mathcal{CI}(G|L)\leqq(G:H)[H:e]$. But $G|L$ is the regular representation of G and thus $\mathcal{CI}(G|L)=\{G:e\}$.

REMARK.[2] If H is a normal subgroup of G and L is the one-dimensional identity representation of H, then $G|L$ contains exactly $\{G:H\}$ distinct irreducible representations of G. Indeed it is sufficient to note that $G|L$ is the composition of the natural projection of G on G/H and the regular representation of G/H. The following proposition gives a sufficient condition for $\{G:H\}$ to be an upper bound to the number of distinct irreducible representations appearing in the decomposition of $G|L$, where L is any linear representation of H. The referee conjectures that $\{G:H\}$ is such an upper bound whenever H is a normal subgroup of G.

PROPOSITION 2. *Suppose H is a normal subgroup of G such that each class of H is also a class of G. Then for every linear representation L of H, $G|L$ contains at most $\{G:H\}$ distinct irreducible representations.*

[2] We are indebted to the referee for this remark.

PROOF. The projection of each class of G onto G/H is contained in a class of G/H. Suppose D_1 and D_2 are two classes of G whose projections on G/H are contained in the same class of G/H. Then the projections of D_1 and D_2 on G/H have nonempty intersection. Thus there exists $x\in D_1$, $y\in D_2$ and $h\in H$ such that $x=hy$.

Note that under our hypothesis $(G|L)_{ghg^{-1}}=L_hI$ for all $g\in G$, where I is the identity operator on $\mathfrak{K}(G|L)$. Indeed for all g, $z\in G$, and $f\in\mathfrak{K}(G|L)$ we have $(G|L)_{ghg^{-1}}f(z)=f(zghg^{-1})=L_{zghg^{-1}z^{-1}}f(z)=L_hf(z)$, where we have used the fact that L is constant on the classes of H.

Let n_i denote the number of elements in the class D_i, for $i=1, 2$. Then we have

$$\begin{aligned} S_1 = \sum_{z\in D_1}(G\,|\,L)_z &= \frac{n_1}{[G:e]}\sum_{g\in G}(G\,|\,L)_{gxg^{-1}} \\ &= \frac{n_1}{[G:e]}\sum_{g\in G}(G\,|\,L)_{ghg^{-1}}(G\,|\,L)_{gyg^{-}} \\ &= \frac{n_1}{[G:e]}L_h\sum_{g\in G}(G\,|\,L)_{gyg^{-}} \\ &= \frac{n_1}{n_2}L_h\sum_{z\in D_2}(G\,|\,L)_z \\ &= \left(\frac{n_1}{n_2}L_h\right)S_2. \end{aligned}$$

Thus $\{G:H\}$ is an upper bound for the number of linearly independent conjugate sums S_i and thus also for the number of distinct irreducible representations appearing in the decomposition of $(G|L)$.

For the theorem to have significance it is necessary to show that $(G:H)$ is indeed a better upper bound than those already known, namely $\{G:e\}$ and $[G:H]$. Let G be the symmetric group on 4 letters. Let H denote the normal abelian subgroup of G of order 4. Then all the numbers associated with the embedding of H in G are distinct. Indeed $[G:H]=6$, $\{G:e\}=5$, $(G:H)=4$ and $\{G:H\}=3$.

It would be interesting to know if the embedding number $(G:H)$ has any significance in any other context than in the theory of group representations which are induced from characters.

REFERENCE

1. J. A. Ernest, *Central intertwining numbers for representations of finite groups*, Trans. Amer. Math. Soc. 99 (1961), 499–508.

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