ON THE CLASSIFICATION OF PERIODIC FLOWS

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1. **Introduction.** Let (S, Σ, μ) be a probability space,[2] and let ϕ and ϕ' be measure-preserving transformations of (S, Σ, μ), that is, functions from Σ to Σ such that $\phi(E)\in\Sigma$ if $E\in\Sigma$ and $\mu(\phi(E))=\mu(E)$. One of the central questions of ergodic theory is: when are ϕ and ϕ' conjugate? We recall that two measure-preserving transformations ϕ and ϕ' are said to be (algebraically) conjugate (cf. Halmos [**2**, p. 44]) when $\phi'=\alpha\circ\phi\circ\alpha^{-1}$ for some transformation α such that both α and its inverse α^{-1} are measure-preserving transformations. We are considering here, for greater generality, *set* transformations of the Boolean σ-algebra Σ, which may or may not be induced by *point* transformations of the set S (cf. Halmos [**2**, pp. 42–45]).

To answer this question, several conjugacy invariants of measure-preserving transformations have been introduced, such as the spectrum of the induced unitary operator, entropy, etc. However, these invariants are at present very far from being a complete set, which would allow an immediate decision on whether or not two measure-preserving transformations are conjugate.

Another approach to the question consists in relating a measure-preserving transformation to some one among a set of "canonical forms," selected for their simplicity. The result of Halmos and von Neumann (cf. Halmos [**2**, p. 48]) goes far in this direction: it is shown that an ergodic measure-preserving transformation with discrete spectrum is conjugate to a rotation on a compact abelian group.

The result of the present note lies in this second direction: we shall prove that *every* periodic group of measure-preserving transformations, or "flow," ergodic or not, is conjugate to a "restriction" of the simplest such flow, namely, the flow obtained on the lateral surface of unit area of a cylinder by rotating the cylinder around its axis.

2. **Definitions.** A *periodic flow* (or simply *flow*) is a semigroup ϕ_t, $0\leqq t\leqq\infty$, of transformations ϕ_t of Σ into Σ, such that $\mu(\phi_t(E))=\mu(E)$ for every E in Σ and for every t, $0\leqq t\leqq\infty$. We shall assume the semigroup property $\phi_t(\phi_s(E))=\phi_{t+s}(E)$, and the periodicity condition $\phi_0=\phi_1$. Clearly, some sort of smoothness condition relative to t of

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[2] We follow the notation in Dunford-Schwartz [1]. Thus S is a set, Σ is a σ-field (or σ-algebra) of "measurable" subsets of Σ, and μ is a measure defined on sets of Σ such that $\mu(S)=1$.

the flow must be assumed. We shall make the weakest such assumption, as follows. Consider the measure space (C, Σ_1, ν) obtained by taking the product of the given measure space (S, Σ, μ) with the interval $[0, 1]$, endowed with ordinary Lebesgue measure. A point of C can be written in the form (t, s), where $0 \leqq t \leqq 1$ and $s \in S$. For $E \in \Sigma$, let E_1 be the set $\{(t, s) \mid s \in \phi_t(E), 0 \leqq t \leqq 1\}$. Our smoothness assumption is that E_1 shall be a measurable set, namely, that $E_1 \in \Sigma_1$, for every $E \in \Sigma$. In other words, as t runs from 0 to 1, $\phi_t(E)$ describes a measurable set in the product.

The above description of a periodic flow can be summarized as follows: a periodic flow is a measurable *action* of the circle group on a probability space. Any continuous action of the circle group on any topological space which is also endowed with a finite regular measure will satisfy our requirements; it is thus easy to produce "many" examples of periodic flows, of the most disparate varieties.

It follows from the definition that ϕ_t^{-1} is well-defined, and since $\phi_t \circ \phi_{1-t} = I$, that $\phi_t^{-1} = \phi_{1-t}$. Defining $\phi_t = \phi_{1-t}$ for $t < 0$, we can obtain a group defined for $-\infty < t < \infty$.

Two flows ϕ_t and ϕ_t', defined on *different* measure spaces (S, Σ, μ) and (S', Σ', μ'), are (algebraically) *conjugate* when $\phi_t = \alpha \circ \phi_t' \circ \alpha^{-1}$, where α is a measure-preserving transformation of Σ onto Σ' with measure-preserving inverse α^{-1}. This implies in particular that the measure-algebras (S, Σ, μ) and (S', Σ', μ') are isomorphic.

A σ-subalgebra Σ'' of Σ is said to be *invariant* under a flow ϕ_t on (S, Σ, μ), whenever $\phi_t(E) \in \Sigma''$ for every $E \in \Sigma''$ and for every t. If Σ'' is an invariant σ-subalgebra of ϕ_t, then clearly one obtains from ϕ_t a new flow ϕ_t'' on the probability space (S, Σ'', μ) by *restricting* ϕ_t to the σ-subalgebra Σ''. The question stated at the end of §1 can now be reformulated as follows: what other flows can one obtain (to within conjugacy) from a given flow ϕ_t, by restricting ϕ_t to each one of its proper invariant σ-subalgebras? If the restriction ϕ_t'' of a flow ϕ_t to some one of its invariant σ-subalgebras Σ'' is conjugate to another flow ϕ_t', we shall say that ϕ_t is a *model* for ϕ_t'.

3. Our main result is then the following

THEOREM. *Let (S, Σ, μ) be a separable nonatomic probability space. Then there exists a universal periodic flow Φ_t in (S, Σ, μ), namely, a flow which is a model for every other periodic flow.*

PROOF. Recall the definition of a nonatomic separable measure-algebra, as given in Halmos [3, pp. 168–173]. From the discussion in Halmos, the following facts are easily derived: (a) the measure algebras of any two nonatomic separable probability spaces are isomorphic; (b) the product of two (and, in fact, of a countably infinite number of) nonatomic separable probability spaces is again a nonatomic separable probability space. (c) In particular, if (S, Σ, μ) is as in the statement of the Theorem, then the measure algebra of the "cylinder" space (C, Σ_1, ν) is isomorphic to that of (S, Σ, μ). We shall write a point of C as the pair (t, s), where $s \in S$ and $-\infty < t < \infty$, agreeing to identify two points whose t-coordinates differ by an integer. We now define Φ_t' on (C, Σ_1, ν) as follows: set $\Phi_t'(E_1) = \{(u, s) \mid (u+t, s) \in E_1\}$. It is obvious that Φ_t' is a flow. Let Φ_t be a flow conjugate to Φ_t' and defined on (S, Σ, μ). Such a flow exists by (c). We claim that Φ_t is the desired universal flow.

To show this, it suffices to show that Φ_t' is a universal flow, or, more simply, by (a), that Φ_t' is a *model* for every flow ϕ_t on (S, Σ, μ).

Given ϕ_t, consider the family Σ'' of sets $E'' = \{(t, s) \mid s \in \phi_t(E), E \in \Sigma\}$. It is clear that Σ'' is a σ-subalgebra of Σ_1, and since Σ is nonatomic, so is Σ''. Furthermore, $\nu(E'') = \mu(E)$. Indeed, $\nu(E'')$ can be calculated by Fubini's theorem, using the fact that the measure ν is the product of Lebesgue measure dt with the measure μ. This gives $\nu(E'') = \int_0^1 \int_S \phi_t(E) d\mu dt$. The inner integral gives $\mu(\phi_t(E)) = \mu(E)$, by the measure-preserving property of ϕ_t. Thus the double integral simplifies to $\nu(E'') = \int_0^1 \mu(\phi_t(E)) dt = \int_0^1 \mu(E) dt = \mu(E)$.

Now define $\alpha: \Sigma \to \Sigma''$ as $\alpha(E) = E''$. Then α is an isomorphism of the measure-algebra of (S, Σ, μ) with that of (C, Σ'', ν). Let Φ_t'' be the restriction of Φ_t' to Σ''. To complete the proof of the theorem, we must establish that Σ'' is an invariant σ-subalgebra of Φ_t' and that the diagram

$$
\begin{array}{ccc}
(C, \Sigma'', \nu) & \xrightarrow[\Phi_t'']{} & (C, \Sigma'', \nu) \\
\alpha \uparrow\downarrow \alpha^{-1} & & \alpha^{-1} \downarrow\uparrow \alpha \\
(S, \Sigma, \mu) & \xrightarrow[\Phi_t]{} & (S, \Sigma, \mu)
\end{array}
$$

is commutative. But, for $E \in \Sigma$, we have by definition

$$
\begin{aligned}
\Phi_t''(\alpha(E)) &= \Phi_t''(E'') = \{(u, s) \mid (u + t, s) \in E''\} \\
&= \{(u, s) \mid s \in \phi_{u+t}(E)\} = \{(u, s) \mid s \in \phi_u(\phi_t(E))\}.
\end{aligned}
$$

This shows that $\Phi_t''(\alpha(E)) = \alpha(\phi_t(E))$, or $\alpha^{-1} \circ \Phi_t'' \circ \alpha = \phi_t$, and completes the proof.

4. **Comments.** The preceding result can be considered as a "strict sense" analog of a result obtained in the author's note [6]. It is

clear that the result can be generalized to any nonseparable but homogeneous measure algebra. However, the statement is significant within the context of separable measure spaces, because it amounts to a restriction on the conjugacy classes of periodic flows.

We conclude with a number of questions suggested by the present situation, which would contribute to the solution of some current problems in ergodic theory.

(1) Suppose ϕ_t is a model for ϕ_t' and ϕ_t' is a model for ϕ_t. What can be said about ϕ_t and ϕ_t'? Are they conjugate? If not, as is probable, then the equivalence relation thus obtained should be studied.

(2) What is the structure of the invariant σ-subalgebras of a measure-preserving transformation ϕ_t?

(3) The usual "functorial" considerations suggest that there should be a dual result for a discrete group, that is, for a single measure-preserving transformation and its powers. In this direction, a method used by Oxtoby [**5**, p. 128], taking infinite direct products of probability spaces, is suggestive.

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