GEOMETRICAL PROPERTIES OF EQUIPOTENTIAL SURFACES

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Let E be a convex set in R^p, $d\mu$ a positive bounded measure on E, $\Phi(r)$ a decreasing function of $r>0$, which is twice continuously differentiable and summable near 0, and $V(M)=\int\Phi(r_{MP})d\mu_P$ the potential of $d\mu$ with respect to Φ at the point M ($M\in R^p$). Our purpose is to describe some properties of the "equipotential surfaces" (which are "curves" for $p=2$)

$$(S_\lambda)=\{M \text{ such that } V(M) = \lambda\}$$

in relation with E, and to generalize some results of J. L. Walsh [1] which concern the case $p=2$, $\Phi(r)=\log 1/r$.

THEOREM 1. *Given $M\notin E$ and $\lambda=V(M)$, let (N) be the normal to (S_λ) at the point M. Then (N) intersects E.*

THEOREM 2. *Suppose $\Phi(r)$ is convex, and let N be the point of $E\cap(N)$ nearest to M. Then, in the neighbourhood of M, (S_λ) does not intersect the open ball of centre N and radius NM.*

THEOREM 3. *Suppose moreover $\Phi''(r)/\Phi'(r)\geqq-(\alpha+1)/r$ $(\alpha\geqq0)$; for example, $\Phi(r)=\log 1/r$ if $\alpha=0$, or $\Phi(r)=r^{-\alpha}$ if $\alpha>0$. Then, in the neighbourhood of M, (S_λ) does not intersect any open ball of radius $\leqq NM/(\alpha+1)$ tangent to (S_λ) at M. Moreover, if E is compact and if the distance d between (S_λ) and E is larger than $(\alpha+2)^{-1/2}\Delta$, Δ being the diameter of E, then (S_λ) is a convex surface.*

The proofs are quite elementary. Let us write $\Phi(r)=\phi(r^2)$. Then $dV(M)=2\boldsymbol{N}\cdot\boldsymbol{dM}$ with

$$\boldsymbol{N}=\int \boldsymbol{PM}\phi'(PM^2)d\mu_P.$$

Now (N) contains Q defined by

$$\boldsymbol{N}=\boldsymbol{QM}\int\phi'(PM^2)d\mu_P.$$

As $\phi'\leqq0$ and E is convex, we have $Q\in E$, which proves Theorem 1.

Let us consider now a curve (C) on (S_λ) through M, such that the center of curvature C at M lies on (N). The conclusion of Theorem 2

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expresses that C is not between N and M. Let s be the arc length and $M(s)$ the current points on (C), $\boldsymbol{t}=d\boldsymbol{M}(s)/ds$, $\boldsymbol{n}/R=d\boldsymbol{t}/ds$ with $|\boldsymbol{n}|=1$; for $s=s_0$, we get $M(s_0)=M$ and $\boldsymbol{MC}=\boldsymbol{n}R$; all calculations below are made at s_0. From $\boldsymbol{t}\cdot\boldsymbol{N}\equiv 0$ there results

$$\frac{\boldsymbol{n}}{R}\cdot\boldsymbol{N}+\boldsymbol{t}\cdot\frac{d\boldsymbol{N}}{ds}=0$$

with

$$\frac{d\boldsymbol{N}}{ds}=\boldsymbol{t}\int\phi'(PM^2)d\mu_P+2\int\boldsymbol{PM}\phi''(PM^2)(\boldsymbol{PM}\cdot\boldsymbol{t})d\mu_P.$$

Therefore

$$\text{(1)}\qquad \frac{(MQ)^-}{(MC)^-}=\frac{\boldsymbol{n}\cdot\boldsymbol{MQ}}{R}=1+2\,\frac{\int(\boldsymbol{PM}\cdot\boldsymbol{t})^2\phi''(PM^2)d\mu_P}{\int\phi'(PM^2)d\mu_P}.$$

If Φ is convex, so is ϕ; therefore

$$\text{(2)}\qquad \frac{(MQ)^-}{(MC)^-}\leqq 1$$

and C does not belong to the interval MQ, which proves Theorem 2.

The assumption $\Phi''(r)/\Phi'(r)\geqq-(\alpha+1)/r$ can be written as $\phi''(r^2)/\phi'(r^2)\geqq-(\alpha+2)/2r^2$. Majorising $(\boldsymbol{PM}\cdot\boldsymbol{t})^2$ by $\boldsymbol{PM}^2$ and Δ^2 respectively, we get from (1)

$$\text{(3)}\qquad \frac{(MQ)^-}{(MC)^-}\geqq -1-\alpha$$

and

$$\text{(4)}\qquad \frac{(MQ)^-}{(MC)^-}\geqq 1-\frac{\alpha+2}{d^2}\Delta^2.$$

(3), together with (2), proves the first part of Theorem 3, and (4) the second one.

BIBLIOGRAPHY

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