UPPER AND LOWER BOUNDS OF THE NORM OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Consider the differential system

$$z' = f(x, z) \tag{1}$$

under the assumptions:

(i) x is a real variable, z and f are finite dimensional complex vectors with n components z_i and f_i respectively,

(ii) f is continuous in (x, z) for all z and for all x in $a \leqq x \leqq b$. Define $|z| = \sum_{i=1}^{n} |z_i|$. Then we have the following

THEOREM 1. *Let the function* $g(x, u) \geqq 0$ *be continuous in the region* $a \leqq x \leqq b$, $u \geqq 0$. *Let the function* $f(x, z)$ *of* (1) *satisfy the condition*

$$|f(x, z)| \leqq g(x, |z|).$$

Let $z(x)$ *satisfy* $|z(x)| > 0$ *and be a solution of* (1) *in the region* $a \leqq x \leqq b$. *Then for all* x *in* $a \leqq x \leqq b$, *we have*

$$|z(x)| \leqq M(x) \tag{2}$$

and

$$|z(x)| \geqq m(x) \tag{3}$$

where $M(x)$ *and* $m(x)$ *are the maximal and minimal solutions of* $u' = \pm g(x, u)$, $u(a) = |z(a)|$, *respectively.*

PROOF. The inequality (2) follows from the Theorem 1 in [4]. To prove (3), we have to use essentially the same argument as in [4] but now we have to consider the minimal solution of $u' = -g(x, u)$, $u(a) = |z(a)|$ instead of the maximal solution of $u' = g(x, u)$, $u(a) = |z(a)|$. This completes the proof.

REMARK. The above theorem includes the results of Bellman [1], Bihari [2] and Langenhop [3] as special cases. Taking $g(x, u) = v(x)u$, it is easy to see that $M(x) = u(a)\exp[\int_a^x v(s)ds]$ and $m(x) = u(a) \exp[-\int_a^x v(s)ds]$, which correspond to Bellman's results. Suppose $g(x, u) = v(x)g(u)$, where $g(u) > 0$ for $u > 0$. Then we can easily obtain $M(x) = G^{-1}[G(u(a)) + \int_a^x v(s)ds]$ and

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$$m(x) = G^{-1}\left[G(u(a)) - \int_a^x v(s)ds\right],$$

where $G(u) = \int_{u_0}^u [g(r)]^{-1}dr$, $u_0 \geqq 0$, which are exactly the results of Langenhop. This implies that the monotonicity assumption regarding $g(u)$ in his hypotheses is superfluous.

We can formulate an interesting comparison theorem. Consider another differential system

$$z' = h(x, z) \tag{4}$$

under the assumptions:

(i) x is a real variable, z and h are finite dimensional complex vectors with n components z_i and h_i respectively,

(ii) h is continuous in (x, z) for all z and for all x in $a \leqq x \leqq b$.

THEOREM 2. *Let the function $g(x, u) \geqq 0$ be continuous in a region $a \leqq x \leqq b$, $u \geqq 0$. Let the functions $f(x, z)$ and $h(x, z)$ satisfy the condition*

$$|f(x, z_1) - h(x, z_2)| \leqq g(x, |z_1 - z_2|).$$

Let the functions $z_1(x)$ and $z_2(x)$ be solutions of (1) *and* (4) *respectively and satisfy $|z_1(x) - z_2(x)| > 0$. Then for all x in $a \leqq x \leqq b$, we have*

$$|z_1(x) - z_2(x)| \leqq M(x); \qquad |z_1(x) - z_2(x)| \geqq m(x),$$

where $M(x)$ and $m(x)$ are the maximal and minimal solutions of $u' = \pm g(x, u)$; $u(a) = |z_1(a) - z_2(a)|$, respectively.

The proof is similar to that of the Theorem 1 and hence omitted.

(Professor R. P. Boas informed me that essentially the same results by the same methods have been obtained independently by A. D. Zeibur.)

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