A STABILITY CRITERION FOR HILL'S EQUATION[1]

HARRY HOCHSTADT

It is well known [1] that the Hill's equation

$$y'' + p(t)y = 0$$

with $p(t)$ periodic with period T, has either two linearly independent bounded solutions, or at least one unbounded solution. In the former case the solutions are said to be stable. Lyapunov [2] proved the following sufficient condition for stability, for the case where $p(t) \geqq 0$:

$$T\int_0^T p(t)dt \leqq 4.$$

Several other criteria have been found and are discussed in reference [1].

This article will consider the case, where $p(t)$ is an even, positive, periodic and differentiable function. One can define two solutions y_1, y_2 by the initial conditions

$$\begin{aligned} y_1(0) &= 1, \qquad & y_2(0) &= 0, \\ y_1'(0) &= 0, \qquad & y_2'(0) &= 1. \end{aligned}$$

From the general theory of Hill's equation [3] one can show that if

$$\begin{aligned} y_1(t) &\geqq 0, \\ y_2'(t) &\geqq 0, \qquad \text{for } 0 \leqq t \leqq T/2, \end{aligned}$$

then all solutions must be bounded.

The following theorem will now be proved.

THEOREM. *A sufficient condition for the boundedness of all solutions of*

$$y'' + p(t)y = 0,$$

where $p(t)$ is an even, positive, differentiable function of period T, is that

$$\int_0^{T/2} (p(t))^{1/2}dt + \frac{1}{4}\int_0^{T/2} \left|\frac{p'(t)}{p(t)}\right| dt \leqq \pi/2.$$

Received by the editors May 15, 1961.

[1] This research was supported by the Air Force Office of Scientific Research, under Contract No. AF 49(638)229 at New York University. Reproduction in whole or in part is permitted for any purpose of the U. S. Government.

The proof is based on the fact that the solutions y_1 and y_2 can be represented as

$$y_1 = A_1(t)\cos\phi_1(t), \qquad y_2 = A_2(t)\sin\phi_2(t),$$
$$y_1' = -(p(t))^{1/2}A_1(t)\sin\phi_1(t), \qquad y_2' = (p(t))^{1/2}A_2(t)\cos\phi_2(t).$$

These representations have been discussed and used in a general analysis of the Sturm-Liouville spectrum [4]. A direct calculation shows that

$$\phi_\nu' = (p(t))^{1/2} + (-)^\nu \frac{1}{4}\frac{p'(t)}{p(t)}\sin 2\phi_\nu$$
$$A_\nu' = -A\frac{p'(t)}{2p(t)}\begin{pmatrix}\sin\\ \cos\end{pmatrix}\phi_\nu\Big)^2,$$

and

$$A_\nu(0) = 1, \qquad \phi_\nu(0) = 0, \qquad \nu = 1, 2.$$

The A have to be exponentials and therefore cannot vanish. Therefore the condition that y_1 and y_2' are non-negative in the interval $[0, T/2]$, is equivalent to

$$|\phi_\nu| \leqq \pi/2 \quad \text{in} \quad [0, T/2], \qquad \nu = 1, 2.$$

Thus we obtain the sufficient condition

$$\begin{aligned}|\phi_\nu(T/2)| &= \left|\int_0^{T/2}\phi_\nu'\,dt\right| \\ &= \left|\int_0^{T/2}(p(t))^{1/2}dt + (-)^\nu\frac{1}{4}\int_0^{T/2}\frac{p'(t)}{p(t)}\sin 2\phi_\nu dt\right| \\ &\leqq \int_0^{T/2}(p(t))^{1/2}dt + \frac{1}{4}\int_0^{T/2}\left|\frac{p'(t)}{p(t)}\right|dt \leqq \frac{\pi}{2}\,.\end{aligned}$$

One can observe that for the case where $p(t)$ is a positive constant, say λ, Lyapunov's criterion yields

$$\lambda \leqq 4/T^2,$$

whereas the present criterion yields

$$\lambda \leqq \pi^2/T^2,$$

which is the best possible estimate.

REFERENCES

1. L. Cesari, *Asymptotic behavior and stability problems in ordinary differential equations*, Springer-Verlag, Berlin, 1959.

2 A. Lyapunov, *Problème général de la stabilité der movement*, Toulouse Univ., Fac. Sci. Ann. (2) 9 (1907), 203–474. Reprinted by Princeton Univ. Press.

3. W. Magnus, and A. Shenitzer, *Hill's equation*. I. *General theory*, New York Univ., Inst. Math. Sci., Div. EM Res., Research Report No. BR-22 (1957).

4. H. Hochstadt, *Asymptotic estimates for the Sturm-Liouville spectrum*, New York Univ., Inst. Math. Sci., Div. EM Res., Research Report No. BR-36; to be published, Comm. Pure Appl. Math. (1961).

NEW YORK UNIVERSITY

COMPARISON THEOREMS FOR LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

WALTER LEIGHTON[1]

1. In this paper we consider self-adjoint differential equations of the form

$$[r(x)y']' + p(x)y = 0, \tag{1}$$

where $r(x)$ and $p(x)$ are continuous and $r(x)>0$ on an interval $\alpha<x<\beta$. By rewriting a theorem in the calculus of variations in a form which emphasizes the behavior of solutions of the Euler-Jacobi equation rather than that of the functional which gives rise to it we are led to observe that the theorem provides a completely general comparison theorem for equations of the form (1). We show that the Sturm and Sturm-Picone[2] theorems may be regarded as special cases of this theorem and incidentally provide in the process useful generalizations of these theorems.

We associate with (1) the functional

$$I = \int_a^b (ru'^2 - pu^2)dx,$$

where the closed interval $[a, b] \subset (\alpha, \beta)$. If $u(x)$ and $r(x)u'(x)$ are functions of class C' on $[a, b]$ and if $u(a)=u(b)=0$, we shall say that

Received by the editors April 24, 1961 and, in revised form, June 29, 1961.

[1] Work on this paper was done while the author received support from the National Science Foundation under grant NSF-G11741.

[2] See, for example, Bôcher [1, p. 53], Ince [2, p. 225].