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REFERENCES

1. R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, Vol. II, Springer, Berlin, 1937.
2. N. Dunford and J. Schwartz, *Linear operators*, Interscience, New York, 1959.
3. P. Halmos, *Measure theory*, Van Nostrand, New York, 1950.
4. E. Hopf, *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachr. **4** (1951), 213–231.
5. M. Riesz, *Sur les ensembles compacts de fonctions sommables*, Acta. Sci. Math. Szeged **6** (1933), 136–142.

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ON THE BASIS PROBLEM IN NORMED SPACES[1]

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Since Schauder's fundamental question of the paper [**9**]—Does every separable Banach space have a basis?—is still unanswered, we give in this note a proof that every infinite-dimensional Banach space has a closed linear subspace which has a basis with certain extra properties. Banach [**1**] asserts without proof the existence of a subspace with a basis. Gelbaum [**4**] proves the existence of a subspace which has a basis with slightly weaker extra properties than those proved in this note. The present theorem was stated, with a few words about the proof, in my book [**3**, p. 72], but repeated requests for copies of a paper in which proof is given suggest that that outline is too obscure.

Indeed, when this note was first sent off, the referee discovered that the proof, given there as sketched in [**3**], was incorrect. We give here a proof based on a generalization of the Borsuk-Ulam theorem. The other "proof" misquoted a theorem of E. Michael in an attempt to reduce the generalization to the original theorem.

THEOREM. *Let N be an infinite-dimensional normed space. Then there exist biorthogonal sequences $(b_i, i \in \omega)$ and $(\beta_i, i \in \omega)$ in B and B^* respectively, such that*

(i) *(b_i) is a basis for the closed linear manifold L in B spanned by the set of all b_i.*

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(ii) $\|b_i\| = \|\beta_i\| = 1$ *for all* i *in* ω.

(iii) *Setting* $P_m x = \sum_{i \leq m} \beta_i(x) b_i$ *for each* m *in* ω, *the linear operator* P_m *is a projection in* L *of norm* $\leqq 1 + 1/m$.

PROOF. Take b_1 of norm 1 in B and choose β_1, by the Hahn-Banach theorem, such that $\|\beta_1\| = \|b_1\| = \beta_1(b_1) = 1$. If $b_1, \cdots, b_m$ in B and $\beta_1, \cdots, \beta_m$, and, if necessary, certain auxiliary $\gamma_1, \cdots, \gamma_k$ in B^* have been chosen, the choice of b_{m+1} is made to depend on Borsuk's theorem in the following way:

Let L_m be the linear hull of the b_i, $i \leqq m$, and let S_m be the unit sphere $S \cap L_m$. If K is the closed real interval $-1 \leqq t \leqq 1$, each of the sets $\beta_i^{-1}(K) \cap L_m$ and $\gamma_j^{-1}(K) \cap L_m$ contains S_m, and the intersection of these sets is a polyhedron Π_m in L_m. It may happen that Π_m contains only points of norm $\leqq 1 + 1/m$; if this is not the case take enough elements of norm 1 in L_m^*, say $\alpha_1, \cdots, \alpha_n$, that Π_m intersected with all the sets $\alpha_q^{-1}(K)$ is a polyhedron in L_m which all lies within the sphere of radius $1 + 1/m$. Let γ_{k+q} be an element of B^* of norm 1 which is an extension of α_q, $q = 1, \cdots, n$. Let Λ_m be the (infinite-dimensional) intersection of the hyperplanes $\beta_i^{-1}(0)$, $i \leqq m$, and $\gamma_j^{-1}(0)$, $j \leqq k + n$.

In Λ_m choose any $(m+1)$-dimensional subspace Λ_m' and in Λ_m' consider the unit sphere $S_m' = S \cap \Lambda_m'$. Define a mapping ϕ from S_m' into convex compact subsets of L_m by

$$\phi(x') = \{y : y \in L_m \text{ and } \|x' + y\| = \|x' + L_m\|\}$$

where, as usual,

$$\|x' + L_m\| = \inf \{\|x' + z\| : z \in L_m\}.$$

Then $\phi(-x) = -\phi(x)$ and ϕ is upper semicontinuous; that is, if x_n' tends to x', if $y_n \in \phi(x_n')$, and $y = \lim_{n \in \omega} y_n$ exists, then $y \in \phi(x')$. Also $\|y\| \leqq 1 + 1/m$ if $y \in \phi(x)$.

To prove that an x exists in S_m' such that $0 \in \phi(x)$, we generalize the Borsuk-Ulam theorem in precisely the same way that Kakutani [7] generalized the Brouwer fixed-point theorem. Jaworowski [6] has a generalization of Borsuk's Theorems I and II which implies this (his $\phi(x)$ is acyclic compact) but rather than appeal to that homological proof we give here a direct proof by approximation using the Borsuk-Ulam theorem for most of the work.

THEOREM. *Let* E *be a compact convex subset of* L_m *and let* ϕ *be a function defined on* S_m' *such that, for each* x, $\phi(x)$ *is a closed convex subset of* E. *Assume that* ϕ *is upper semicontinuous* (*or equivalently, that the graph of* ϕ *is closed*) *and that* ϕ *is antipodal; that is, that* $\phi(-x) = -\phi(x)$. *Then there is an* x *in* S_m' *such that* $0 \in \phi(x)$.

Let Σ_n be a sequence of "simplicial" decompositions of S'_m into "spherical triangles" with the diameters of the simplexes of Σ_n tending to zero as $n\to\infty$, and let $x_{ni} \cdots x_{nk}$ be the vertices of Σ_n in some order. Define a continuous function ϕ_n from S'_m into E by first choosing $\phi_n(x_{ni})$ as an element of $\phi(x_{ni})$, choosing $\phi_n(-x_{ni}) = -\phi_n(x_{ni})$, and then extending the map to all S'_m simplicially. Each ϕ_n is continuous and antipodal, so the Borsuk-Ulam theorem asserts that there is a point p_n of S'_m such that $\phi_n(p_n)=0$.

(In [2, Theorem II] it is asserted only that some pair of antipodal points is carried to a single image point; but ϕ_n maps antipodal pairs to pairs symmetric about 0; these can coincide if and only if they both are zero.)

p_n is in some simplex of Σ_n with vertices $y_{n0}, \cdots, y_{nm}$, so $p_n = \sum_{i=0}^m \lambda_{ni} y_{ni}$, where $\lambda_{ni} \geqq 0$, $\sum_i \lambda_{ni} \geqq 1$, and $\lim_n \sum_i \lambda_{ni} = 1$. Choose a subsequence $\{n_j\}$ of the integers so that $y_i = \lim_j y_{n_j i}$ exists, $z_i = \lim_j \phi_{n_j}(y_{n_j i})$ exists, and $\lambda_i = \lim_j \lambda_{n_j i}$ exists for $i=0, 1, \cdots, m$. Then $p = \lim_j p_{n_j}$ exists also and $p = y_i$ for all i. Also $\sum_i \lambda_i = 1$. Now

$$0 = \phi_{n_j}(p_{n_j}) = \phi_{n_j}\left(\sum_i \lambda_{n_j i} y_{n_j i}\right) = \sum_i \lambda_{n_j i} \phi_{n_j}(y_{n_j i}) \to \sum_i \lambda_i z_i.$$

But $z_i \in \phi(p)$ because ϕ is upper semicontinuous. Hence $0 = \sum \lambda_i z_i \in \phi(p)$ because $\phi(p)$ is convex.

To return now to the main theorem, let b_{m+1} be any point of S'_m such that $0 \in \phi(b_{m+1})$. Let β_{m+1} in B^* be chosen so that it is of norm 1, vanishes on the b_i, $i \leqq m$, and is 1 at b_{m+1}. This is possible because the calculation with Borsuk's theorem arranged matters so that in L_{m+1} the projection along L_m onto the line through b_{m+1} is of norm 1. Also Λ_m was chosen so that the projection of $\Lambda_m + L_m$ along Λ_m onto L_m is of norm $\leqq 1+1/m$. This induction process defines sequences (b_i) and (β_i).

If L' is the union of all the L_m, then for each m the function P_m is defined in B and has in L' norm $\leqq 1+1/m$, because all the b_i, $i>m$, are in Λ'_m. Since L' is dense in L, $\|P_m\|_L \leqq 1+1/m$. The set of those x in L where $\lim_{m\in\omega} P_m x = x$ includes all of the b_i and is closed in L [1, p. 79, Theorem 3], so it is all of L; that is, (b_i) is a basis for L, and the proof is complete.

Note that if in (iii) the condition $\|P_m\| \leqq 1+1/m$ were replaced by $\|P_m\| \leqq 1+c_m$, where (c_m) is any preassigned bounded sequence of positive numbers, the proof would be almost unaltered.

It may be noted that if this process is applied to $m(\omega)$, the space of all bounded real sequences, nothing in the process prevents the choice of b_n equal to the usual nth basis vector in $c_0(\omega)$, the subspace of sequences converging to zero; that is, $b_n(n)=1$, $b_n(m)=0$ if $m\neq n$. Then the corresponding sequence of β's may be total over $m(\omega)$, and the process stops even though $m(\omega)$ is not separable. This example shows that the basis found need not be boundedly complete in the sense of James [5] and Karlin [8]. It also shows, by Sobczyk's result of [10], that there need not be any continuous projection of N onto L. The construction of the theorem if applied in $l^1(\omega)$ might also yield the usual basis, which shows that the biorthogonal system $\{\beta_i\}$ need not be a basis for L^*.

In view of Karlin's theorem that there is no unconditional basis in $C[0, 1]$, the space of real-valued functions continuous on the closed unit interval, it would be of interest to improve this construction to show that every B contains an L with an unconditional basis; but whether this is possible is not known.

Bibliography

1. S. Banach, *Théorie des opérations linéaires*, Monogr. Mat., Vol. 1, Warsaw, 1932.

2. K. Borsuk, *Drei Sätze über die n-dimensional Euklidische Sphäre*, Fund. Math. **20** (1933), 177–190.

3. M. M. Day, *Normed linear spaces*, Springer, Berlin-Göttingen-Heidelberg, 1958.

4. B. R. Gelbaum, *Notes on Banach spaces and bases*, An. Acad. Brasil. Ci. **30** (1958), 29–36.

5. R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. (2) **52** (1950), 518–527.

6. J. W. Jaworowski, *Theorem on antipodes for multi-valued mappings and a fixed point theorem*, Bull. Acad. Polon. Sci. Cl. III, **4** (1956), 187–192.

7. S. Kakutani, *A generalization of Brouwer's fixed point theorem*, Duke Math. J. **8** (1941), 457–459.

8. S. Karlin, *Bases in Banach spaces*, Duke Math. J. **15** (1948), 971–985.

9. J. Schauder, *Zur Theorie stetiger Abbildungen in Funktionalräumen*, Math. Z. **26** (1927), 47–65 and 417–431.

10. A. Sobczyk, *Projection of the space m on its subspace c_0*. Bull. Amer. Math. Soc. **47** (1941), 938–947.

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