SELF-ADJOINT TOEPLITZ OPERATORS AND ASSOCIATED ORTHONORMAL FUNCTIONS[1]

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1. **Introduction.** Let l^2 be the Hilbert space of square summable sequences $f=(f_0, f_1, f_2, \cdots)$. l^2 is isomorphic to the space H^2 of functions holomorphic in the unit disk Δ with square integrable boundary values, under the map

$$(1.1)\qquad f \to F, \qquad F(z) = \sum_{n=0}^{\infty} f_n z^n.$$

A Toeplitz operator is an l^2 linear operator T to which corresponds a function W on the unit circle Γ, such that under the isomorphism (1.1) we have for the inner product

$$(1.2)\qquad \langle Tf, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(\theta)F(e^{i\theta})G^*(e^{i\theta})d\theta.$$

Here * denotes complex conjugation.

This work is concerned with the concrete spectral theory of Toeplitz operators that are associated with functions W that satisfy the following two hypotheses:

(i) W is real, bounded below, and absolutely integrable on Γ, but it is not equivalent to a constant function.

(ii) For each real λ the set $\Gamma_\lambda = \{\theta: W(\theta) \leqq \lambda\}$ is, modulo a set of measure zero, an arc of the circle.

By concrete spectral theory we mean[2] that we exhibit an explicit sigma-finite measure ρ on $(-\infty, \infty)$ and an explicit unitary correspondence $U: l^2 \to L^2(d\rho)$ such that $UTU^{-1} = M$, where M is the multiplication operator on $L^2(d\rho)$ which sends $g(\lambda)$ into $\lambda g(\lambda)$. Hypothesis (i) implies that T is bounded below, so its Friedrichs extension (again named T) is self-adjoint.

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The results obtained are conveniently described in terms of the set of vectors $k(u) \in l^2$ defined for each $u \in \Delta$ by

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[2] This description applies only to operators with simple spectrum. The hypotheses given guarantee that T has simple spectrum, but this is not in general true for Toeplitz operators, cf. [6].

$$(1.3) \qquad k_n(u) = u^n, \qquad k_0(0) = 1.$$

Under the correspondence (1.1) it follows that $K(u;\, z) = (1-uz)^{-1}$, and thus for each $f \in l^2$ we have

$$(1.4) \qquad \langle f,\, k(u^*)\rangle = F(u).$$

Let E be the spectral measure of T. In Theorem 3 we find that there is an absolutely continuous measure ρ on the line given by (3.3) and a collection of functions $\Phi(u;\, \lambda)$ given by (3.2) such that for all $u,\, v \in \Delta$ and each real Borel set Λ

$$(1.5) \qquad \langle E(\Lambda)k(u),\, k(v)\rangle = \int_\Lambda \Phi(u;\lambda)\Phi^*(v,\,\lambda)d\rho(\lambda).$$

According to (2.7) and Lemma 1 the functions $\Phi(u;\lambda)$ are for almost all λ holomorphic functions of $u \in \Delta$ with the Maclaurin expansion

$$(1.6) \qquad \Phi(u;\,\lambda) = \sum_{n=0}^{\infty} \phi_n(\lambda)u^n.$$

For fixed $u \in \Delta$, $\Phi(u;\, \lambda) \in L^2(d\rho)$. Now the transformation $k(u) \to \Phi(u;\, \lambda)$ defined on the set $\mathcal{K} = \{k(u) : u \in \Delta\}$ in l^2 with range in $L^2(d\rho)$ preserves inner products as can be seen by taking $\Lambda = (-\infty,\, \infty)$ in (1.5). Since $\mathcal{K}$ is total in l^2, cf. (1.4), it follows that there exists a unique isometry $U: l^2 \to L^2(d\rho)$ such that $Uk(u) = \Phi(u;\lambda)$. This transformation has the explicit form below obtained from (1.3) and (1.6):

$$(1.7) \qquad Uf = \sum_{0}^{\infty} f_n\phi_n.$$

We next note that (1.5) implies that for each real Borel set Λ, U sends $E(\Lambda)k(0)$ into the product of the indicator function of Λ and $\Phi(0;\, \lambda)$. Since by (3.2) $\Phi(0;\, \lambda)$ is almost everywhere nonzero, it follows that the range of U is $L^2(d\rho)$. Thus U is a unitary mapping of l^2 onto $L^2(d\rho)$.

As a corollary of the fact that (1.7) is a unitary equivalence we conclude that $\{\phi_n\}$ is a complete orthonormal set in $L^2(d\rho)$. So we see that $\Phi(u;\lambda)$ is a generating function for a complete orthonormal set of functions in $L^2(d\rho)$. In the examples at the end we specify W so as to obtain certain Gegenbauer and Pollaczek polynomials.

2. **Analysis of Toeplitz matrices.** Suppose now that hypothesis (i) is satisfied, and set $\lambda_0 = \text{ess inf } W$. It is known, [1], that whenever $\lambda < \lambda_0$ there is a factorization

$$(2.1) \qquad W(\theta) - \lambda = |H_\lambda(e^{i\theta})|^2,$$

where H_λ is an outer function in H^2, i.e., the set $\{z^n H_\lambda(z)\}_{n=0}^{\infty}$ is total in H^2. H_λ is uniquely specified if we impose the normalization $H_\lambda(0) > 0$. The explicit formula is

$$H_\lambda(u) = \exp \int_{-\pi}^{\pi} \log (W(\theta) - \lambda) P^*(u^*, \theta) d\theta, \qquad u \in \Delta \tag{2.2}$$

where

$$P(u, \theta) = \frac{1}{4\pi} (1 + ue^{i\theta})(1 - ue^{i\theta})^{-1}.$$

Thus whenever $\lambda < \lambda_0$ and T is given by (1.2) we may write $T - \lambda = S_\lambda^* S_\lambda$, where S_λ is the operator which under the map (1.1) transforms into multiplication by H_λ, i.e.

$$S_\lambda f \to H_\lambda F. \tag{2.3}$$

From (1.4) we easily compute that

$$S_\lambda^* k(u) = H_\lambda^*(u^*) k(u). \tag{2.4}$$

Since H_λ is an outer function, S_λ and S_λ^* have densely defined inverses. Hence $(T-\lambda)^{-1} = S_\lambda^{-1} S_\lambda^{*-1}$. Using (2.4) we see that

$$\langle (T - \lambda)^{-1} k(u), k(v) \rangle = H_\lambda^{*-1}(u^*) H_\lambda^{-1}(v^*) \langle k(u), k(v) \rangle. \tag{2.5}$$

It is useful to rewrite this formula using (2.2) and the fact that $\langle k(u), k(v) \rangle = (1 - uv^*)^{-1}$. We obtain

$$\begin{aligned} &\langle (T - \lambda)^{-1} k(u), k(v) \rangle \\ &= (1 - uv^*)^{-1} \exp - \int_{-\pi}^{\pi} \log (W(\theta) - \lambda) [P(u, \theta) + P^*(v, \theta)] d\theta. \end{aligned} \tag{2.5'}$$

For u, $v \in \Delta$, the right hand side of (2.5′) is holomorphic in the λ-plane cut along the real axis from λ_0 to ∞. Thus (2.5′) provides an analytic continuation of the resolvent of T. We will apply the Stieltjes inversion formula

$$\begin{aligned} d\langle E(\lambda) k(u), k(v) \rangle / d\lambda = \frac{1}{2\pi i} \lim_{\epsilon \downarrow 0} \{ &\langle (T - \lambda - i\epsilon)^{-1} k(u), k(v) \rangle \\ &- \langle (T - \lambda + i\epsilon)^{-1} k(u), k(v) \rangle \} \end{aligned} \tag{2.6}$$

to (2.5′), making use of the following

LEMMA 1. *For almost all real* λ

$$\int_{-\pi}^{\pi} \log | W(\theta) - \lambda | d\theta > - \infty.$$

PROOF. Let $z=\lambda+i\epsilon$, $\epsilon>0$, and consider $I(z)=\int_{-\pi}^{\pi}\log(W(\theta)-z)d\theta$. This equals

$$k\int_{-\pi}^{\pi}\log\left[(W(\theta)-z)(1+W(\theta)^2)^{-1/2}\right]d\theta,$$

where $k=(1/2)\int_{-\pi}^{\pi}\log\left[1+W(\theta)^2\right]d\theta$. Thus

$$I(z)=\int_{-\infty}^{\infty}\log\left[(t-z)(1+t^2)^{-1/2}\right]d\alpha(t),$$

where α is bounded and monotone. Upon partial integration we have $I(z)=\int_{-\infty}^{\infty}\left[(t-z)^{-1}-t(1+t^2)^{-1}\right]\alpha(t)dt$. Thus $\lim_{\epsilon\downarrow 0} I(z)$ exists a.e. and $\lim_{\epsilon\downarrow 0}\operatorname{Re} I(z)=\lim_{\epsilon\downarrow 0}\int_{-\pi}^{\pi}\log\left|W(\theta)-\lambda+i\epsilon\right|d\theta$ is finite a.e. The lemma now follows from this by monotone convergence.

The lemma allows us to conclude that

$$\Psi(u;\lambda)=\exp-\int_{-\pi}^{\pi}\log\left|W(\theta)-\lambda\right|P(u,\theta)d\theta \tag{2.7}$$

defines for almost all λ a holomorphic function of $u\in\Delta$. Let $\Gamma_\lambda=\{\theta\colon W(\theta)\leqq\lambda\}$. Evidently

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\log(W(\theta)-\lambda\pm i\epsilon)&=\log\left|W(\theta)-\lambda\right|\quad\text{when }\theta\notin\Gamma_\lambda\\&=\log\left|W(\theta)-\lambda\right|\pm\pi i\text{ when }\theta\in\Gamma_\lambda.\end{aligned}$$

We can now calculate the right-hand side of (2.6) from (2.5′). The result is

THEOREM 1. *Let T be a Toeplitz operator defined by* (1.2), *where W satisfies hypothesis* (i). *Then the resolution of the identity $E(\lambda)$ of T satisfies a.e.*

$$d\langle E(\lambda)k(u),k(v)\rangle/d\lambda$$
$$=\pi^{-1}\Psi(u;\lambda)\Psi^*(v;\lambda)(1-uv^*)^{-1}\sin\left\{\pi\int_{\Gamma_\lambda}\left[P(u,\theta)+P^*(v,\theta)\right]d\theta\right\}.$$

Theorem 1 describes the absolutely continuous part of T. That this is a complete description follows from

THEOREM 2. *Let T be a Toeplitz operator defined by* (1.2) *where W satisfies hypothesis* (i). *Then the spectral resolution of T is weakly absolutely continuous with respect to Lebesgue measure.*

PROOF. See [6]. Theorem 2 makes use of the assumption that W is not equivalent to a constant. The two theorems combined show that the spectrum of T is purely continuous and consists of the closed interval [ess inf W, ess sup W]. This result is known, cf. [**3** and **4**]. What remains to be done is to exploit Theorem 1. This will require the use of hypothesis (ii).

3. **Concrete spectral theory.** We shall need the simple computational

LEMMA 2. *If* $0 \leqq b-a \leqq 2\pi$,

$$\int_a^b P(u,\theta)d\theta = \frac{1}{4\pi}(b-a) + \frac{1}{2\pi i}\log[(1-ue^{ia})(1-ue^{ib})^{-1}].$$

Hypothesis (i) says that $\Gamma_\lambda = \{a(\lambda) \leqq \theta \leqq b(\lambda)\}$ where $0 \leqq b(\lambda) - a(\lambda) \leqq 2\pi$. From Lemma 2 we have

$$\begin{aligned} &\pi \int_{\Gamma_\lambda} [P(u,\theta) + P^*(v,\theta)]d\theta \\ (3.1)\quad &= \frac{1}{2}(b-a) + (2i)^{-1}\log[(1-ue^{ia})(1-v^*e^{-ib})(1-ue^{ib})^{-1}(1-v^*e^{-ia})^{-1}] \end{aligned}$$

where $a = a(\lambda)$ and $b = b(\lambda)$. The main theorem is

THEOREM 3. *Let T be a Toeplitz operator defined by* (1.2) *where W satisfies hypotheses* (i) *and* (ii). *Then in the spectral decomposition of T we have for each real Borel set Λ*

$$(1.5)\qquad \langle E(\Lambda)k(u), k(v)\rangle = \int_\Lambda \Phi(u;\lambda)\Phi^*(v;\lambda)d\rho(\lambda)$$

where

$$(3.2)\qquad \Phi(u;\lambda) = \Psi(u;\lambda)(1-ue^{ia(\lambda)})^{-1/2}(1-ue^{ib(\lambda)})^{-1/2}$$

and

$$(3.3)\qquad d\rho(\lambda) = \pi^{-1}\sin\frac{1}{2}[b(\lambda)-a(\lambda)]d\lambda.$$

PROOF. With the aid of (3.1) the sine term in the formula of Theorem 1 can be calculated. We obtain

$$d\langle E(\lambda)k(u), k(v)\rangle/d\lambda = \Phi(u;\lambda)\Phi^*(v,\lambda)\rho'(\lambda)$$

for almost all λ where $\rho'(\lambda) = d\rho(\lambda)/d\lambda$. Theorem 2 asserts that the set function $\langle E(\cdot)k(u), k(v)\rangle$ is absolutely continuous, whence the assertion of Theorem 3 follows.

COROLLARY 1. *The functions $\Phi(u;\lambda)$ are for almost all λ holomorphic functions of $u \in \Delta$ with the Maclaurin expansion* (1.6). *The mapping*

$U: \{f_n\} \rightarrow \sum_0^\infty f_n\phi_n$ is a unitary transformation of l^2 onto $L^2(d\rho)$ such that $UTU^{-1} = M$, where $M: g(\lambda) \rightarrow \lambda g(\lambda)$.

PROOF. See the introduction.

4. Examples.

The results obtained are:

EXAMPLE 1. $W(\theta) = \cos\theta$.

$$\begin{aligned} \rho'(\lambda) &= \pi^{-1}(1-\lambda^2)^{1/2}, && |\lambda| < 1 \\ &= 0, && |\lambda| \geqq 1. \\ \Phi(u;\lambda) &= 2^{1/2}(1-2\lambda u+u^2)^{-1}, \\ \phi_n(\lambda) &= 2^{1/2}C_n^{(1)}(\lambda), \quad \text{where} \quad C_n^{(1)}(\lambda) \end{aligned}$$

is the nth Gegenbauer polynomial of order 1, cf. [2, p. 174].

EXAMPLE 2. $W(\theta) = \sin\theta$. $\rho'(\lambda)$ is as in Example 1.

$$\begin{aligned} \Phi(u;\lambda) &= 2^{1/2}(1-2i\lambda u-u^2)^{-1}, \\ \phi_n(\lambda) &= 2^{1/2}(-i)^n C_n^{(1)}(\lambda). \end{aligned}$$

EXAMPLE 3. $W(\theta) = 1$ if $|\theta| \leqq c$; $= 0$ otherwise. Here $0 < c < \pi$.

$$\rho'(\lambda) = \frac{1}{\pi}\sin c \text{ if } 0 < \lambda < 1, = 0 \text{ otherwise.}$$

Set $\beta = -1/2\pi \log(\lambda^{-1}-1)$. Then

$$\begin{aligned} \Phi(u;\lambda) &= e^{c\beta}(1+e^{-2\pi\beta})^{1/2}(1-ue^{-ic})^{-1/2-i\beta}(1-ue^{ic})^{-1/2+i\beta}, \\ \phi_n(\lambda) &= e^{c\beta}(1+e^{-2\pi\beta})^{1/2}P_n^{(1/2)}(\beta, c), \end{aligned}$$

where $P_n^{(1/2)}(\beta, c)$ is the nth Pollaczek polynomial, cf. [5], of order $1/2$ for $(-\infty, \infty)$.

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