A NOTE ON COHOMOLOGY ADDITION

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Professor A. D. Wallace proposed in a Research Problem a question concerning the validity of a cohomology addition theorem [3]. We shall answer his question in the negative. We shall only use the AWS-cohomology groups and the singular homology groups. First a pair of definitions with the coefficient group suppressed. Let X be a Hausdorff space and $h \in H^n(X)$ a nonzero cohomology class, then X is a minimal support for h if and only if for every proper closed subset $C \subset X$ the class h lies in the kernel of $i^*: H^n(X) \to H^n(C)$. Let $A \subset X$ be a closed subset and $h \in H^n(A)$ a nonzero class. A closed set $M \subset X$ is an irreducible membrane for h if and only if for every proper closed subset $C \subset M$ the class h lies in the image of $i^*: H^n(C \cup A) \to H^n(A)$, but is not in the image of $i^*: H^n(M \cup A) \to H^n(A)$. The two concepts are developed extensively in (1). We now state the problem. Let $X = X_1 \cup X_2$ be a compact Hausdorff space expressed as the union of two closed subsets. If $X_1 \cap X_2$ is a minimal support for some nonzero cohomology class $h \in H^n(X_1 \cap X_2)$ and if X_1 and X_2 are both irreducible membranes for h, does it then follow that $H^{n+1}(X) \neq 0$? We shall show by example that it does not follow. Two points should be noted. First, the question does bear some resemblance to the well-known homology addition theorem, and second if $X = X_1 \cup X_2$ is a finite simplicial complex expressed as the union of two closed subcomplexes then it follows from the remaining hypotheses, with field coefficients, that $H^{n+1}(X) \neq 0$. It is quite possible that the use of field coefficients is unnecessary in this case.

Now for the example. Let $K \subset S^2$ be a subcontinuum of the 2-sphere which irreducibly separates it, and for which $S^2 - K = C_1 \cup C_2 \cup C_3$ is the union of three disjoint open connected sets each of which is homeomorphic to the open 2-cell. Such a continuum is presented in [2, p. 118]. For each component C_i, the frontier $F(C_i) = K$. Let $X_i = C_i^* = C_i \cup K$ for $i = 1, 2, 3$. By the Alexander duality theorem $\tilde{H}^j(K; Z_2) \simeq \tilde{H}_{1-j}(S^2 - K; Z_2)$, thus $H^1(K; Z_2) \simeq Z_2 + Z_2$. If $A \subset K$ is a proper closed subset, then $H^1(A; Z_2) = 0$, for otherwise K would not irreducibly separate S^2. We conclude that K is a minimal support for every nontrivial class in $H^1(K; Z_2)$.

The pair (X_i, K) is a relative 2-cell, thus $H^2(X_i, K; Z_2) \simeq Z_2$, and $H^j(X_i, K; Z_2) = 0$, $j \neq 2$. From the exact sequence of pair (X_i, K) we have

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$$0 \to H^1(X_i; Z_2) \to H^1(K; Z_2) \to H^2(X_i, K; Z_2) \to 0$$

from which we see $H^1(X_i; Z_2) \simeq Z_2$ and $i^*: H^1(X_i; Z_2) \to H^1(K; Z_2)$ is a monomorphism. For each $i = 1, 2, 3$ there is a unique nonzero cohomology class $e_i \in H^1(K; Z_2)$ which lies in the image of $i^*: H^1(X_i; Z_2) \to H^1(K; Z_2)$.

Let $X = X_1 \cup X_2$, then X is a proper subcontinuum of S^2 which fails to separate, thus $H^j(X; Z_2) = 0$, $j > 0$. From the Mayer-Vietoris sequence of $X = X_1 \cup X_2$ we see that $I^*: H^1(X_1; Z_2) + H^1(X_2; Z_2) \simeq H^1(K; Z_2)$. This shows $e_1 \neq e_2$, but even more the three classes e_1, e_2, e_3 are distinct and $e_1 + e_2 = e_3$.

Finally we must show that if $h \in H^1(K; Z_2)$ is not in the image of $i^*: H^1(X_i; Z_2) \to H^1(K; Z_2)$ then X_i is an irreducible membrane for h. In any case there is an irreducible membrane for h, $M \subset X_i$ such that $M \supset K$ [1, 1.3, 1.4]. If $M \neq X_i$, then M omits an interior point of X_i. Now the interior of X_i is C_i, an open 2-cell. Around any point in C_i we may select a small regular open 2-cell $\sigma \subset C_i$ so that $F(\sigma) \cap K = \emptyset$ and $F(\sigma) = S^1$. The space obtained from $X_i - \sigma$ by collapsing K to a point is a closed 2-cell, thus from excision $H^j(X_i - \sigma, K; Z_2) = 0$, $j \geqq 0$ and $i^*: H^1(X_i - \sigma; Z_2) \simeq H^1(K; Z_2)$. We now see that the irreducible membrane $M \subset X_i$ cannot omit any interior point of X_i, thus $M = X_i$, an irreducible membrane for h.

We note that $X_1 \cap X_2 = K$ is a minimal support for $e_1 + e_2$, while X_1, X_2 must both be irreducible membranes for $e_1 + e_2$, however $H^2(X; Z_2) = 0$.

References

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