ON A THEOREM BY I. GLICKSBERG

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If X is a compact space we denote by $C(X)$ the Banach space of all continuous complex valued functions on X with respect to the norm $\|f\| = \sup_{x\in X} |f(x)|$. Grothendieck [2, Theorem 5] has shown that a bounded subset of $C(X)$ is compact in the weak topology if and only if it is compact in the topology of pointwise convergence on X.

Using an extension of this theorem I. Glicksberg [1, Theorem 1.2] recently proved

THEOREM 1. *Let X and Y be locally compact Hausdorff spaces, and f a bounded complex function on $X\times Y$ which is separately continuous, i.e., for which all the maps*

$$x \to f(x, y) \quad \text{and} \quad y \to f(x, y)$$

are continuous. Then for μ a bounded Radon measure on Y,

$$x \to \int f(x, y) d\mu(y)$$

is continuous.

The purpose of this note is to prove the following more general

THEOREM 2. *Let X be a locally compact and Y a locally compact Hausdorff space. Let f be a complex function on $X\times Y$ which is separately continuous, and μ a Radon measure on Y. Suppose furthermore that $|f(x, y)| \leqq g(y)$ for all $x\in X$ and almost all $y \in Y$ (i.e., for all $y\in Y$ except a μ-null set), where $g \in L^1(\mu)$. Then*

$$x \to \int f(x, y) d\mu(y)$$

is continuous.

PROOF. The proof could be derived from Theorem 1. However, we prefer to derive it directly from Grothendieck's theorem by a slight modification of Glicksberg's proof.

Let ϵ be any positive number. Since g is zero outside the union of a μ-null set and a countable set of compact sets, and every compact set

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$K \subset Y$ contains a compact set K_1 such that $\mu(K-K_1)$ is arbitrarily small and the restriction of g to K_1 is continuous, there exists a compact set $K \subset Y$ such that the restriction of g to K is continuous, and such that

$$\int_{Y-K} g(y)d\mu(y) < \epsilon.$$

We may furthermore assume that K is such that

$$|f(x, y)| \leqq g(y) \text{ for all } x \in X \text{ and all } y \in K.$$

Since g is continuous on K, there exists a constant M such that $|f(x, y)| \leqq M$ for all $x \in X$ and all $y \in K$.

Let now a be any point of X and (x_α) be any generalized sequence (i.e., directed set) of points in X which converges to a, and V be a compact neighborhood of a. Denote by f_x the restriction of the mapping $y \rightarrow f(x, y)$ to K. Then $A = \{f_x\}$, $x \in V$, is a bounded subset of $C(K)$ which is compact in the topology of pointwise convergence on X, since the mapping $x \rightarrow f_x$ of X into $C(K)$ endowed with the topology of pointwise convergence on K is continuous. Hence by Grothendieck's theorem A is compact in the weak topology of $C(K)$ and the two topologies agree on A since they are comparable. It follows that (f_{x_α}), $x_\alpha \in V$, converges weakly to f_x. Hence there exists an α such that

$$\left| \int_K [f(a, y) - f(x_\beta, y)] d\mu(y) \right| < \epsilon \qquad \text{for } \beta > \alpha.$$

Therefore

$$\begin{aligned} &\left| \int [f(a, y) - f(x_\beta, y)] d\mu(y) \right| \\ &\leqq \left| \int_K [f(a, y) - f(x_\beta, y)] d\mu(y) \right| + 2 \int_{Y-K} g(y) d\mu(y) < 3\epsilon \quad \text{for } \beta > \alpha, \end{aligned}$$

which proves that the mapping $x \rightarrow \int f(x, y) d\mu(y)$ is continuous at a.

References

1. I. Glicksberg, *Weak compactness and separate continuity*, Pacific J. Math. **11** (1961), 205–214.

2. A. Grothendieck, *Critères de compacité dans les espaces fonctionnels généraux*, Amer. J. Math. **74** (1952), 168–186.

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