ON A RESULT OF HADAMARD CONCERNING THE SIGN OF THE PRECESSION OF A HEAVY SYMMETRICAL TOP

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Consider Lagrange's integrable case of the motion of a rigid body about a fixed point, when the center of gravity of the body lies on the polar axis of the spheroid of inertia of the body [1, case II, pp. 216–249].

In [1, §110, pp. 233–235] one finds "An interesting proof that the precession for a complete period has the same sign as ω_k (which) has been given by Hadamard" (see [2]). Hadamard's proof employs the theory of residues of functions of a complex variable. Quite recently (see [3], which cites the recent literature), upper and lower bounds for the apsidal angle in the theory of the heavy symmetrical top have been obtained by an elementary method, *not* relying upon the theory of residues. This same direct method applies equally well in deducing Hadamard's result. This application will be carried out here, in the notation of [1], for the immediate comparison of the two methods.

The argument requires the preliminary evaluation of two simple definite integrals:

$$\pi = \int_{u_3}^{u_2} \frac{((1+u_2)(1+u_3))^{1/2}du}{(1+u)((u_2-u)(u-u_3))^{1/2}}$$

$$= \int_{u_3}^{u_2} \frac{((1-u_2)(1-u_3))^{1/2}du}{(1-u)((u_2-u)(u-u_3))^{1/2}}, \qquad -1 < u_3 < u_2 < 1.$$

Just put $u = -v$ in the second integral to obtain the first; and put $1+u = (1+u_3)(1+u_2)(1+v^2)[(1+u_2)+(1+u_3)v^2]^{-1}$, $0 \leqq v < \infty$, in the first.

The problem [1, p. 233] is to show that the definite integral

$$\frac{a^{1/2}}{b\omega_k}(\psi - \psi_0) = \int_{u_3}^{u_2} \frac{u_4 - u}{1-u^2} \frac{du}{((u_1-u)(u_2-u)(u-u_3))^{1/2}},$$

$$-1 < u_3 < u_4 < u_2 < 1 < u_1,$$

is positive, the square root being positive. The case under consideration, where u_4 satisfies $u_3 < u_4 < u_2$, is that in which "loops" occur [1, Figure 61, p. 242], and is the only "doubtful" case; if $u_4 \geqq u_2$ or $u_4 \leqq u_3$ then the state of affairs is evident.

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Now, since

$$\frac{u_4 - u}{1 - u^2} = \frac{1}{2}\left[\frac{1 + u_4}{1 + u} - \frac{1 - u_4}{1 - u}\right],$$

it follows that

$$\begin{aligned} 2\int_{u_3}^{u_2} \frac{u_4 - u}{1 - u^2} \frac{du}{((u_1 - u)(u_2 - u)(u - u_3))^{1/2}} \\ = \int_{u_3}^{u_2} \frac{1 + u_4}{1 + u} \frac{du}{((u_1 - u)(u_2 - u)(u - u_3))^{1/2}} \\ - \int_{u_3}^{u_2} \frac{1 - u_4}{1 - u} \frac{du}{((u_1 - u)(u_2 - u)(u - u_3))^{1/2}}. \end{aligned}$$

However,

$$\frac{1}{(u_1 + 1)^{1/2}} < \frac{1}{(u_1 - u)^{1/2}} < \frac{1}{(u_1 - 1)^{1/2}}, \quad -1 < u_3 \leqq u \leqq u_2 < 1 < u_1.$$

Hence

$$\begin{aligned} 2\frac{a^{1/2}}{b\omega_k}(\psi - \psi_0) \\ > \pi\left[\frac{1 + u_4}{((u_1 + 1)(1 + u_2)(1 + u_3))^{1/2}} - \frac{1 - u_4}{((u_1 - 1)(1 - u_2)(1 - u_3))^{1/2}}\right] \\ = 0; \end{aligned}$$

since, from [1, p. 221], when $u_3 < u_4 < u_2$,

$$\frac{1 - u_4}{1 + u_4} = \left(\frac{(u_1 - 1)(1 - u_2)(1 - u_3)}{(u_1 + 1)(u_2 + 1)(u_3 + 1)}\right)^{1/2}.$$

References

1. W. D. MacMillan, *Dynamics of rigid bodies*, Dover, New York, 1960.

2. J. Hadamard, *Sur la précession dans le mouvement d'un corps pesant de révolution fixé par un point de son axe*, Bull. Sci. Math. 19 (1895), 228–230.

3. J. B. Diaz and F. T. Metcalf, *Upper and lower bounds for the apsidal angle in the heory of the heavy symmetrical top* (submitted to the Journal of Mathematical Physics).

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