ON THE RATIONAL TRIANGULATION OF A CIRCLE

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A triangle is called *rational* if its sides and area are rational. A set of rational triangles is referred to as a *rational triangulation of a circle* K if (a) each triangle is inscribed in K, (b) no two of the triangles have interior points in common, and (c) the sum of the areas of the triangles is equal to the area of K.

THEOREM. *A circle can be rationally triangulated if and only if its radius is rational.*

PROOF. If a, b, c are the lengths of the sides of a triangle inscribed in a circle of diameter k, then the area of the triangle is

$$abc/2k. \tag{1}$$

Thus, it is impossible to inscribe a rational triangle in a circle having irrational radius.

The case where K has a rational radius is now considered. Let K be given by

$$r = k \cdot \cos \theta \qquad (k \text{ rational}). \tag{2}$$

Let R be the set of all numbers θ $(-\pi/2 < \theta \leqq \pi/2)$ such that both $\cos \theta$ and $\sin \theta$ are rational, and S the set of all points (r, θ) on (2) such that $\theta \in R$. The following properties of S will be established:

(i) *a point of S can be selected on any subarc of* (2);

(ii) *any three points of S are the vertices of a rational triangle*;

(iii) *on any minor subarc $\widehat{AC}$ of* (2), *a point $B \in S$ can be selected such that the area of $\triangle ABC$ exceeds one-fourth that of the segment $\widehat{ACA}$* (*i.e. the segment of* (2) *bounded by the arc $\widehat{AC}$ and the chord AC*).

PROOF OF (i). Let $\widehat{P_1P_2}$ be any subarc of (2), where P_1: $(k \cdot \cos \alpha_1, \alpha_1)$, P_2: $(k \cdot \cos \alpha_2, \alpha_2)$, $-\pi/2 < \alpha_1 < \alpha_2 \leqq \pi/2$. If $\alpha_1 < 0 < \alpha_2$, then the point $(k, 0)$ satisfies (i). In each of the remaining cases, $-\pi/2 < \alpha_1 < \alpha_2 \leqq 0$ and $0 < \alpha_1 < \alpha_2 \leqq \pi/2$, which are now considered, it is noted that $\cos \alpha_1 \neq \cos \alpha_2$. Let the (unordered) set $\{\cos \alpha_1, \cos \alpha_2\}$ be denoted by $\{\lambda, \mu\}$ where $0 \leqq \lambda < \mu \leqq 1$. A rational number W can be selected such that

$$\left(\frac{1-\mu}{1+\mu}\right)^{1/2} < W < \left(\frac{1-\lambda}{1+\lambda}\right)^{1/2}.$$

Then

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(3) $$\lambda < \frac{1 - W^2}{1 + W^2} < \mu.$$

Let

$$\phi = \begin{cases} -\operatorname{Arccos}\dfrac{1 - W^2}{1 + W^2} & \text{if } \alpha_1 < \alpha_2 \leqq 0, \\ +\operatorname{Arccos}\dfrac{1 - W^2}{1 + W^2} & \text{if } 0 < \alpha_1 < \alpha_2. \end{cases}$$

Then $\phi \in R$, and thus the point Q: $(k \cdot \cos \phi, \phi) \in S$. Since $\cos \phi$ lies between $\cos \alpha_1$ and $\cos \alpha_2$ (cf. (3)) and the function $\cos x$ is monotonic on $-\pi/2 < x \leqq 0$ and on $0 < x \leqq \pi/2$, the point Q is on $\widehat{P_1P_2}$.

PROOF OF (ii). Let O denote the pole, and P_1: $(k \cdot \cos \theta_1, \theta_1)$, P_2: $(k \cdot \cos \theta_2, \theta_2)(\theta_1 < \theta_2)$ be any two points in S. Since angle P_1OP_2 equals $(\theta_2 - \theta_1)$, $|P_1P_2| = k \cdot \sin(\theta_2 - \theta_1)$. Thus the distance between any two points in S is rational. Consequently, any triangle having vertices in S has rational sides, and, since k is rational, rational area (cf. (1)). Therefore (ii) holds.

PROOF OF (iii). Let $ACPQ$ be the rectangle circumscribed about $\widehat{ACA}$ (i.e. PQ is tangent to $\widehat{AC}$ at its midpoint). Let M and N be the midpoints of AQ and CP, respectively, and denote the points of intersection of MN with $\widehat{AC}$ by A' and C'. A point $B \in S$ can be selected on the minor arc $\widehat{A'C'}$ (cf. (i)). The area of $\triangle ABC$ exceeds that of $\triangle AMC$, namely, one-fourth the area of the rectangle $ACPQ$. Since the area of the rectangle $ACPQ$ exceeds that of the segment about which it is circumscribed, (iii) holds.

Let T be the set of all triangles having elements of S as vertices; every element of T is a rational triangle (cf. (ii)). It will now be established that (2) can be rationally triangulated by a subset of T.

In view of (i) and (ii) an acute triangle $T_0 \in T$ can be selected. In each minor segment of (2) subtended by a side XY of T_0, a point $Z \in \widehat{XY} \cap S$ can be selected so that $\triangle XYZ \in T$ has area greater than one-fourth that of $\widehat{XYX}$ (cf. (iii)); thus triangles $T_i \in T$ $(i = 1, 2, 3)$ are obtained. Operating similarly on each segment of (2) subtended by a side of the boundary of $T_0 \cup T_1 \cup T_2 \cup T_3$ (considering T_i as closed regions), triangles $T_i \in T$ $(i = 4, 5, \cdots, 9)$ are obtained; operating similarly on $T_0 \cup \cdots \cup T_9$, triangles $T_i \in T$ $(i = 10, 11, \cdots, 21)$ are obtained; etc.

Thus, if t_i denotes the area of T_i $(i = 0, 1, 2, \cdots)$, and $u_j = 3(2^j - 1)$, then

$$\frac{\pi k^2}{4} - \sum_{i=0}^{u_n} t_i < \frac{3}{4}\left(\frac{\pi k^2}{4} - \sum_{i=0}^{u_{n-1}} t_i\right) \qquad (n = 1, 2, 3, \cdots).$$

Consequently,

$$\frac{\pi k^2}{4} - \sum_{i=0}^{u_n} t_i < \left(\frac{3}{4}\right)^n \left(\frac{\pi k^2}{4} - t_0\right),$$

and hence,

$$\lim_{m\to\infty} \sum_{i=0}^{m} t_i = \frac{\pi k^2}{4}.$$

Thus, the set $\{T_0, T_1, T_2, \cdots\}$ is a rational triangulation of (2). This completes the proof of the theorem.

REMARKS. (1) The triangulation $\{T_0, T_1, T_2, \cdots\}$ is *locally finite* in the sense that any circle inside (2) intersects only a finite number of the T_i. It is possible to construct rational triangulations of (2) which are *not* locally finite.

(2) A "rational refinement" may be obtained by joining the midpoints of the sides of each T_i. Another rational refinement (and extension) technique is given in [1].

As an immediate consequence of the theorem, a result is obtained apropros the following generally unresolved problem: If $C_1, C_2, \cdots, C_n$ are n circles in a plane, is it possible to find points $P_1, P_2, \cdots, P_n$ inside $C_1, C_2, \cdots, C_n$, respectively, such that the distance between P_i and P_j (for all i, j) is rational? (L. J. Mordell has answered this question in the affirmative for $n=4$ (cf. [2]).)

COROLLARY. *If a circle K passes through interior points of each of the n circles $C_1, \cdots, C_n$, then it is possible to find points $P_1, \cdots, P_n$ inside $C_1, \cdots, C_n$ respectively, such that the distance between P_i and P_j (for all i, j) is rational.*

PROOF. Since K passes through interior points of each C_i, then it is possible to find a circle K' with rational radius which passes through interior points of each C_i. The corollary is an immediate consequence of (i) and (ii).

REFERENCES

1. F. Supnick, *Rational triangulations*, Amer. Math. Monthly **68** (1961), 95–102.
2. L. J. Mordell, *Rational quadrilaterals*, J. London Math. Soc. **35** (1960), 277–282.

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