NOTE ON CERTAIN COMBINATORIAL DESIGNS[1]

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Let v, μ_1, n and λ_n be integers such that $v > \mu_1 \geqq 1$, $n \geqq 1$ and $\lambda_n \geqq 1$. By a (λ_n, n, μ_1, v) *tactical configuration* is meant an arrangement of v objects into certain blocks such that each block consists of μ_1 distinct objects and that each set of n objects occurs together in exactly λ_n blocks [3, p. 264]. Any (λ_n, n, μ_1, v) tactical configuration with $n > 1$ is a $(\lambda_{n-1}, n-1, \mu_1, v)$ tactical configuration where

$$\lambda_{n-1} = \lambda_n \frac{v - n + 1}{\mu_1 - n + 1}$$

holds[2] [2, p. 16]. In other words, in an arrangement of v objects into certain blocks consisting of μ_1 distinct objects each, if each set of $n > 1$ objects occurs together in exactly λ_n blocks, then each set of $n-1$ objects occurs together in exactly λ_{n-1} blocks, where λ_{n-1} is an integer defined above. A $(\lambda_2, 2, \mu_1, v)$ tactical configuration is called a (λ_2, μ_1, v) *block design*. Fisher's inequality states that the number b of blocks in a (λ_2, μ_1, v) block design is not smaller than v [1, p. 84]. A block design with $b = v$ is called *symmetric*. A $(1, n, \mu_1, v)$ tactical configuration is called a (n, μ_1, v) *Steiner system*. By interchanging the role of objects and that of blocks, we define dual configurations which will be designated by asterisks. Thus, for example, by a $(\mu_n, n, \lambda_1, b)^*$ tactical configuration is meant an arrangement into b blocks of certain objects such that each object occurs in exactly λ_1 blocks and that each set of n blocks has exactly μ_n objects in common. If a (λ_2, μ_1, v) block design is symmetric, then it is also a $(\lambda_2, \mu_1, v)^*$ block design [1, p. 124].

An arrangement of a finite number of objects, called points, into certain blocks, called lines, subject to the following postulates is called a *finite projective plane*:

P_1 Two distinct points are contained in one and only one line.

P_2 Two distinct lines contain one and only one point in common.

P_3 There exists at least one set of four points, no three of which are contained in one line.

A finite projective plane is a symmetric $(1, t+1, t^2+t+1)$ block design for some $t \geqq 2$, where t is called the order of the plane.

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[2] This remark was also made by B. R. Kripke.

In this note we prove some propositions which show that apparently more general conditions can be realized only by tactical configurations which are our old acquaintances, such as finite projective planes, $(v-1, v-1, v)$ Steiner systems.

LEMMA. *Let T be a (λ_n, n, μ_1, v) tactical configuration for some $n \geqq 2$ such that each set of m blocks has exactly $\mu_m = \lambda_n$ objects in common for some $m \geqq 2$. Then $m = n$.*

PROOF. T is a (λ_i, i, μ_1, v) tactical configuration for $i = n, n-1, \cdots,$ 2, 1, and also a $(\mu_i, i, \lambda_1, b)^*$ tactical configuration for $i = m, m-1, \cdots, 2, 1$. Fisher's inequality together with its dual implies that $b = v$. Therefore

$$\lambda_i = \mu_i \quad \text{for } i = 1, 2, \cdots, \mathrm{Min}(n, m).$$

Since $\mu_m = \lambda_n$, we have $m = n$.

PROPOSITION 1. *Let T be a (n, μ_1, v) Steiner system such that each set of m blocks has exactly one object in common for some m and some n. Then $m = n$ and*

(1) *In case $n = 1$, T is the $(1, 1, v)$ Steiner system.*

(2) *In case $n = 2$, T is a finite projective plane of order $\mu_1 - 1$ or the $(2, 2, 3)$ Steiner system according as $\mu_1 \geqq 3$ or $\mu_1 = 2$.*

(3) *In case $n \geqq 3$, T is the $(v-1, v-1, v)$ Steiner system.*

PROOF. Obviously, $n = 1$ if and only if $m = 1$. Hence by the previous lemma $m = n$ always. We need no proof for the case $n = 1$. Suppose that $n = 2$. If P_3 is fulfilled, then T is a finite projective plane of order $\mu_1 - 1 \geqq 2$. If P_3 is violated, then $v \geqq 4$ implies that $\mu_1 \geqq 3$, from which P_3 follows, a contradiction. Hence $v = 3$ and $\mu_1 = 2$. This proves the case $n = 2$. Suppose that $n \geqq 3$. T is a (λ_i, i, μ_1, v) tactical configuration for $i = n, n-1, \cdots, 3, 2, 1$, and also a $(\mu_i, i, \lambda_1, b)^*$ tactical configuration for $i = n, n-1, \cdots, 3, 2, 1$. Fisher's inequality together with its dual implies that $b = v$. Since

$$\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_{n-1} > \lambda_n = 1,$$

we have $\lambda_2 \geqq n-1$. On the other hand, $\mu_n = 1$ implies that $\lambda_2 \leqq n-1$. Hence $\lambda_2 = n-1$ and therefore

$$\lambda_i = n - i + 1 \qquad \text{for } i = 2, 3, \cdots, n-1, n.$$

A dual argument shows that

$$\mu_i = n - i + 1 \qquad \text{for } i = 2, 3, \cdots, n-1, n.$$

Suppose that two distinct blocks can be written in terms of objects by

$$A = \{a_1, a_2, \cdots, a_{n-1}, x, z, \cdots\},$$

and

$$B = \{a_1, a_2, \cdots, a_{n-1}, y, \cdots\},$$

where different letters represent different objects. Since $n \geqq 3$, there exists a block

$$C = \{a_1, a_2, \cdots, a_{n-3}, x, y, z, \cdots\}.$$

If neither a_{n-2} nor a_{n-1} belongs to C, then three distinct blocks A, B and C have exactly $n-3$ objects in common, a contradiction. On the other hand, if either a_{n-2} or a_{n-1} belongs to C, then two distinct blocks A and C have at least n objects in common, a contradiction. This shows that $\mu_1 = n$ and that any two distinct blocks must be of the form

$$A = \{a_1, a_2, \cdots, a_{n-1}, x\},$$

and

$$B = \{a_1, a_2, \cdots, a_{n-1}, y\},$$

where different letters represent different objects. If there exists an object z not included in either A or B, then there exists a block

$$D = \{a_1, a_2, \cdots, a_{n-3}, x, y, z\}.$$

Three distinct blocks A, B and D have exactly $n-3$ objects in common, a contradiction. This shows that two distinct blocks A and B exhaust all objects and that $v = \mu_1 + 1$. This proves the case $n \geqq 3$.

PROPOSITION 2. *Let T be a (λ_n, n, μ_1, v) tactical configuration for some $n \geqq 3$ such that each set of m blocks has exactly μ_m objects in common for some $m \geqq 2$. Then T is the $(v-1, v-1, v)$ Steiner system.*

PROOF. T is a (λ_i, i, μ_1, v) tactical configuration for $i = 3, 2, 1$, where $\lambda_2 > 1$, and also a $(\mu_i, i, \lambda_1, b)^*$ tactical configuration for $i = 2, 1$, where $\mu_2 > 1$. Fisher's inequality gives $b = v$. Let T' be the arrangement obtained from T in the following way: First, drop all blocks which do not contain a specific object a_1. Then, drop the object a_1 from the remaining blocks. T' is a $(\lambda_3, 2, \mu_1 - 1, v-1)$ tactical configuration and also a $(\mu_2 - 1, 2, \lambda_2, \lambda_1)^*$ tactical configuration. Again by Fisher's inequality we have $\lambda_1 = v-1$ and therefore $\mu_1 = v-1$.

PROPOSITION 3. *Let T be an arrangement of v objects into certain blocks such that*

(1) *There exists* $n \geqq 2$ *such that each set of* n *objects occurs together in exactly one block.*

(2) *Any two distinct blocks have exactly* μ_2 *objects in common.*

(3) *For any two blocks there exist* $n-1$ *distinct objects included in neither of these two blocks.*

Then T *is a finite projective plane.*

PROOF. Let A and B be any two distinct blocks and let $a_1, a_2, \cdots, a_{n-1}$ be $n-1$ distinct objects included in neither A nor B. We define an equivalence relation in A by letting x, y equivalent if and only if $x, a_1, a_2, \cdots, a_{n-1}$ and $y, a_1, a_2, \cdots, a_{n-1}$ determine the same block. Do the same in B. The number of objects in any equivalence class is μ_2. Letting an equivalence class in A correspond to an equivalence class in B if and only if the two classes determine the same block together with $a_1, a_2, \cdots, a_{n-1}$, we see that A and B have the same number of equivalence classes and therefore consist of the same number of objects. This shows that every block consists of the same number of objects, say μ_1. Hence T is a (n, μ_1, v) Steiner system for some $n \geqq 2$. Suppose that $n \geqq 3$. Then, by the previous proposition, T is the $(v-1, v-1, v)$ Steiner system. This is impossible by (3). Now, $n=2$ and $\mu_2=\lambda_2=1$. This together with (3) gives $\mu_1 \geqq 3$, from which P_3 follows.

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