ON EXPONENTIALLY CLOSED FIELDS[1]

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It is well known [4] that the non-Archimedean residue class fields K of the ring of continuous real valued functions on a space are real-closed and η_1-sets. It does not appear to be known that the exponential function in the reals induces an exponential function in K (definitions to follow); thus K is exponentially closed. The property of being exponentially closed is a new invariant which will be applied to totally ordered fields in this paper.

A totally ordered field K will be called *exponentially closed* if (i) there exists an order preserving isomorphism f of the additive group of K onto K^+, the multiplicative group of positive elements of K, and (ii) there exists a positive integer n such that $1+1/n<f(1)<n$; such an isomorphism will be called an *exponential function* in K.

In §0 Archimedean exponentially closed fields will be considered, the rest of the paper being devoted to the non-Archimedean case. In §1 some necessary conditions for a non-Archimedean field to be exponentially closed will be given, followed in §2 by some examples. In §3 a set of sufficient conditions will be given, followed by an example.

A totally ordered field K will be called *root-closed* if K^+ is divisible. Clearly exponentially closed fields and real-closed fields are root-closed.

0. An Archimedean totally ordered field is isomorphic to a unique subfield of the reals. Let K be an exponentially closed subfield of the reals and let f be an exponential function in K. If $a=f(1)$ then $f(x)=a^x$ for all $x\in K$. Conversely, if $a\in K$, $a>1$, and if $g(x)$ is defined to be a^x for all $x\in K$, then g is an exponential function in K. Thus, any subfield of the reals is contained in a unique exponentially closed subfield of the reals, both having the same cardinality. The field of real algebraic numbers is, by definition, real-closed. However $2^{2^{1/2}}$ is not in it, hence it is not exponentially closed. It is not known to the author if exponentially closed fields need be real-closed.

1. **Necessary conditions.** Let K be a non-Archimedean field. It is well known [3] that one can associate with K a totally ordered group

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G and a homomorphism V of the multiplicative group of K onto G satisfying the following conditions: (1) V is order preserving on K^+, (2) $V(a \pm b) \leqq \max (V(a), V(b))$ ($V(0)$ being the symbol $-\infty$ treated in the usual way), and (3) $V(a) = V(b)$ if and only if there exists a positive integer n such that $|a| \leqq n|b|$ and $|b| \leqq n|a|$. The mapping V will be called a *natural valuation on K*; clearly any two such mappings are essentially identical. The valuation ring of V is $O = \{a \in K: V(a) \leqq 0\}$ and its maximal ideal $P = \{a \in K: V(a) < 0\}$. Clearly the residue class field of K, $O/P = k$, is an Archimedean field.

Assume, in addition, that K is exponentially closed and that f is an exponential function in K.

LEMMA 1.1. *The restriction of f to O maps O onto the group of positive units of O. Further, $a \in P$ if and only if $f(a) - 1 \in P$.*

PROOF. Since $f(1) < n$, f maps O into the positive units of O. Let a be a positive unit of O. There exists $m \in N$, the set of positive integers, such that $1/m < a < m$. Let $b = f^{-1}(a)$. It suffices to show that $b \in O$. There exists $i \in N$ such that $(1+1/n)^i > m$. Thus $f(i) = f(1)^i > (1+1/n)^i > m > f(b)$, and $i > b$. Since $1+1/n < f(1)$, $f(-1) < n/(n+1)$. Since $n/(n+1) < 1$ there exists $t \in N$ such that $(n/(n+1))^t < 1/m$. Thus $f(-t) = f(-1)^t < (n/(n+1))^t < 1/m < f(b)$ and $-t < b$, proving that $b \in O$ and hence the first assertion is proved.

Let h be the canonical homomorphism of O onto k. Clearly $r = hf$ is an order preserving homomorphism of O onto the multiplicative group of positive units of k. Clearly $a \in P$ if and only if $-1 < ma < 1$ for all integers m. By condition (ii), $1 + 1/n \leqq r(1) \leqq n$ and $1/n \leqq r(-1) \leqq n/(n+1)$. Hence a is in P if and only if $1/n \leqq (r(a))^m \leqq n$ for all integers m: i.e., $r(a) = 1$ or equivalently $f(a) - 1 \in P$, proving the lemma.

The following theorem is an immediate consequence of this lemma.

THEOREM 1.2. *The residue class field of a non-Archimedean exponentially closed field is an Archimedean exponentially closed field.*

The restriction of V to K^+ is an order preserving homomorphism onto G whose kernel is the group of positive units of O; thus Vf is an order preserving homomorphism of the additive group of K onto G whose kernel is O, proving the following theorem.

THEOREM 1.3. *If K is a non-Archimedean exponentially closed field whose valuation ring is O and whose value group is G then there exists an order preserving group isomorphism that sends K/O onto G.*

It is well known [3] that given a totally ordered Abelian group G

there exists a mapping W of G onto a totally ordered set that has all the properties of V, except that of being a homomorphism. Such a mapping, characterized by these properties, will be called a *natural valuation on G*. Let G^+ be the set of positive elements of G. Then $S=W(G^+)$ will be called the *value set of G*. For $s\in S$ let $G_s = \{g\in G: W(g)\leqq s\}/\{g\in G: W(g)<s\}$. Clearly G_s, which will be referred to as the *factor of G associated with s*, is an Archimedean group.

COROLLARY 1.4. *Assume that K is a non-Archimedean exponentially closed field. Let G be the value group of K and k the residue class field of K. Then G^+ is isomorphic as an ordered set to $W(G^+)$ and the factors of G are isomorphic to k.*

PROOF. By Theorem 1.3, K/O and G are isomorphic; thus they have isomorphic value sets. The value set of K/O under the natural valuation induced by V is G^+, proving the first assertion. Let $g\in G^+$. The factor of K/O associated with g is isomorphic to the factor K_g of K associated with g. Let $a\in K$ such that $V(a)=g$. Then $K_g=Oa/Pa$, which is isomorphic to $O/P=k$, proving the corollary.

2. **Examples.** Under pointwise operations, the set $C(X)$ of all continuous functions from a completely regular Hausdorff space into the reals is a lattice-ordered ring. If $a\in C(X)$ then $e^a\in C(X)$; further, a and e^a-1 have the same zeros and hence [4] belong to the same maximal ideals. Let K be a non-Archimedean residue class field of $C(X)$ [4] and let h be the associated canonical homomorphism. For $a'\in K$ let $a\in h^{-1}(a')$, and let $f(a')=h(e^a)$. Since $a'=0$ if and only if $h(e^a-1)=0$, f is a well defined isomorphism of K into K^+. Since h and $a\to e^a$ are order preserving, so then is f. For $a'\geqq 1$ we may choose $a\geqq 1$. Let $b=\log a$ and let $b'=h(b)$. Clearly $f(b')=a'$. For $0<a'<1$ we may apply the argument above to $1/a'$; thus K is exponentially closed.[2]

It is well known [4] that such fields are real-closed, have the reals as their residue class field and are η_1-sets in the sense of the following definition. Let α be an ordinal number and let T be a totally ordered set. T is called an *η_α-set* if, given subsets A and B of T of power less than $\aleph_\alpha$ such that $A<B$, then there exists $t\in T$ such that $A<\{t\}<B$.

It has been shown [2] that if $\alpha>0$, $\aleph_\alpha$ is a regular cardinal number, and $\sum_{\delta<\alpha} 2^{\aleph_\delta}\leqq\aleph_\alpha$, then a real-closed field exists that is an η_α-set of power $\aleph_\alpha$. Let K be such a field. Clearly K is non-Archimedean. Let $f_0(n)=2^n$ for all integers n. Both the additive group of K and the

[2] According to Henriksen, this argument can be used to show that the residue class fields of uniformly closed phi-algebras are exponentially closed.

multiplicative group of positive elements of K are totally ordered Abelian divisible groups that are η_α-sets of power $\aleph_\alpha$. Thus by Theorem B [1] f_0 extends to an exponential function in K, proving that K is exponentially closed.

Let k be an Archimedean field and let T be a nonempty totally ordered set. For $a \in k^T$ let $s(a) = \{t \in T: a(t) \neq 0\}$. A subset of T is called *anti-wellordered* if every nonempty subset of it has a greatest element. Let $k\{T\}$ be defined to be $\{a \in k^T: s(a)$ is anti-wellordered$\}$. Clearly $k\{T\}$ is an Abelian group under pointwise addition. For $a \in k\{T\}$, $a \neq 0$, let $d(a)$ be the greatest element in $s(a)$. Define $a > 0$ if $a(d(a)) > 0$; then $k\{T\}$ is a totally ordered group, d is a natural valuation and T is its value set.

For an ordinal number α let $k\{T\}_\alpha = \{a \in k\{T\}$: the cardinal number of $s(a)$ is less than $\aleph_\alpha\}$. Clearly $k\{T\}_\alpha$ is a subgroup of $k\{T\}$. Let G be a nonzero totally ordered Abelian group. For $a, b \in k\{G\}$ let $(ab)(g) = \sum_{x \in G} a(x)b(g-x)$. It is well known [5] that, under this multiplication, $k\{G\}$ is a totally ordered field. Let α be a nonzero ordinal number; then $k\{G\}_\alpha$ is a subfield of $k\{G\}$. Further, d restricted to $k\{G\}_\alpha$ is a natural valuation of $k\{G\}_\alpha$, its value group being G and its residue class field k.

Let G be a totally ordered Abelian divisible group that is an η_1-set of power $\aleph_1$ and let $K = R\{G\}$. It was shown in [2] that K is a real-closed field that is an η_1-set and has as its residue class field the reals; thus K might be conjectured to be isomorphic to a residue class field of $C(X)$ for some X. However K/O is isomorphic to $R\{G^+\}$ which is of power $2^{\aleph_1}$, whereas G is of power $\aleph_1$; thus, by Theorem 1.3, K is not exponentially closed and hence not isomorphic to any residue class field of $C(X)$ for any space X.

3. **Sufficient conditions.** Let k be an Archimedean field, α a nonzero ordinal, G a nonzero totally ordered Abelian group, and let $K = k\{G\}_\alpha$. The valuation ideal of K is $k\{G^-\}$, G^- being the set of all negative elements of G. It has been shown [5] that given a nonzero element q of P then the semigroup $\omega s(q) (= \bigcup_{n \in N} ns(q))$ of G is anti-wellordered, and further given g in it there exists $m \in N$ such that $g \in \bigcup_{n=1}^m ns(q)$. Thus given a sequence $(a_n)_{n \in N}$ in k, $r = \sum_{n=1}^\infty a_n q^n$ is a well defined element of P. Further, given $b \in K$, $rb = \sum_{n=1}^\infty a_n q^n b$.

For $q \in P$ let $\exp q = \sum_{n=0}^\infty q^n/n!$ and let $\log 1 + q = \sum_{n=1}^\infty (-1)^{n-1} q^n/n$. By direct calculation it is seen that for all $q, r \in P$, $\exp q \exp r = \exp q + r$. From analysis we know that $\sum_{n=1}^\infty (-1)^{n-1} (\sum_{m=1}^\infty x^m/m!)^n/n$ converges for all real x such that $|x| < \log 2$; and further that the sum of this series, since it is the expansion of log e^x, is x. Hence the coefficients of this series are the same as the coefficients of the power series x. Thus log exp $q=q$ for all $q\in P$, proving that exp maps P onto $1+P$ and is one-to-one.

Let K be a non-Archimedean field with value group G and residue class field k. We will say that K is *properly imbedded* in $k\{G\}$ if it is imbedded in $k\{G\}$ such that given $a\in K$, $V(a)=d(a)$, and such that $k\{G\}_0\subset K$. Generalizing somewhat a well known result stated by Conrad [**3**, p. 328] we get the following: if K is real-closed it can be properly imbedded in $k\{G\}$.

THEOREM 3.1. *A non-Archimedean field K with valuation ring O, valuation ideal P, value group G and residue class field k is exponentially closed if the following hold*: (0) *K is root-closed*, (1) *k is exponentially closed*, (2) *K/O is order isomorphic to G, and* (3) *K may be properly imbedded in $k\{G\}$ in such a way that if $q\in P$ then* exp q *and* log $1+q$ $\in K$.

PROOF. Let K be imbedded in $k\{G\}$ such that condition (3) holds; thus exp is an order preserving isomorphism of P onto the multiplicative group of $1+P$. Let $\bar{k}=k1$. Clearly the ring O is the direct sum of $\bar{k}$ and P, the order on the sum being lexicographic. By condition (1), k is exponentially closed; thus given $a\in k$, $a>1$, the mapping $x\rightarrow a^x$ is an exponential in k. For $y\in O$ let $y=x+q$, $x\in\bar{k}$ and $q\in P$, this decomposition being unique. Let $f_0(y)=a^x \exp q$. Clearly f_0 is an order preserving isomorphism of O onto the group of positive units of O. The additive group of K is the direct sum of K/O and O, the order in the sum being lexicographic. An element u in K can be expressed uniquely as $z+y$, $z\in K/O$ and $y\in O$. By condition (2) there exists an order preserving isomorphism t of K/O onto G. Let $f(u)=(t(z), f_0(y))$. The valuation V, restricted to K^+, is an order preserving homomorphism of the multiplicative group of K^+ (which is divisible by condition (0)) onto G whose kernel is the group of positive units of O. Thus the totally ordered group K^+ is the direct product of G and the group of multiplicative units of O, the order being lexicographic. Hence f becomes an exponential function of K, proving that K is exponentially closed, proving the theorem.

Note. Conditions (0), (1) and (2) are necessary for K to be exponentially closed.

Let E be an η_1-set of power $\aleph_1$ and let $(x_n)_{n\in N}$ be a strictly increasing sequence in E. Let $E_n=\{x\in E: x<x_n\}$. Then E_n is an η_1-set of power $\aleph_1$. Let $E'=\bigcup_{n\in N} E_n$. Since E' has a countable cofinal sequence it is not an η_1-set. Let $G=R\{E'\}_1$ and let $G_n=R\{E_n\}_1$. Then G_n is an Abelian divisible group that is an η_1-set of power $\aleph_1$ [**2**]. Further,

$G=\bigcup_{n\in N} G_n$; thus G^+ is order isomorphic to E' which, under the natural valuation d, is the value set of G (cf. Corollary 1.4).

$K=R\{G\}_1$ is a real-closed field (hence a root-closed field) that has the reals as its residue class field; thus K satisfies conditions (0) and (1). K/O is isomorphic to $R\{G^+\}_1$ which, since G^+ is isomorphic to E', is isomorphic to $R\{E'\}_1$: i.e., to G; thus K satisfies condition (2). Clearly condition (3) holds. Thus, by Theorem 3.1, K is exponentially closed. However, since K has a countable cofinal sequence it is not an η_1-set.

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