COMPLETE SETS OF REPRESENTATIONS OF ALGEBRAS

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1. Introduction and results. A classical theorem [1, Chapter XV, Theorem IV] states:

(1) *Let G be a finite group and R a faithful representation*[1] *of G over a field K. Then each irreducible representation of G over K is a constituent of some tensor power of R.*

The only proof of this result known to us actually requires the additional assumption that K is of characteristic 0 and involves a calculation with characters which is not very revealing (to us). In an attempt to construct a more conceptual proof we have been led to a considerably more general result.

(2) *Let A be an algebra over a field K. Assume that A has a basis B over K such that* $B\cup\{0\}$ *is closed under multiplication. Finally, let R be a representation of A which is faithful on* $B\cup\{0\}$, *and for each* $r=1, 2, \cdots$ *let* $\otimes^r R$ *be the representation of A defined by* $(\otimes^r R)(b) = \otimes^r R(b)$ $(b\in B)$ *together with linearity. Then the representations* $\otimes^r R$ $(r=1, 2, \cdots)$ *form a complete set of representations of A (in the sense that their direct sum is faithful on A).*

Observe that the assumptions on B imply that each $\otimes^r R$ really is a representation of A and that A is associative, but that there is no restriction on the characteristic of K or the dimension of A or R. The transition from (2) to (1) is immediately effected by applying to the group algebra of G the statement (2) and the following probably well-known result, for which a proof is sketched at the end of this paper.

(3) *If* $\{{}^rR \mid r=1, 2, \cdots\}$ *is a complete set of representations of a finite-dimensional algebra A, then each irreducible representation of A is a constituent of some* ${}^rR$.

That the finiteness assumptions cannot be dropped in (1) or (3) may be seen from the following example. Let $e(k)$ be the real 2×2 matrix obtained by replacing the 12 entry of the identity matrix by k, G the multiplicative group of all $e(k)$, A the group algebra of G over the reals, B the set G (imbedded in A), and R the defining representation of G extended to A. Then no tensor power of R contains the one-dimensional representation S of A (or G) defined by $S(e(k)) = \exp k$ (k real).

The proof of (2) depends on the following lemma.

Received by the editors September 21, 1961.

[1] Throughout this note all representations are assumed to correspond to left modules and the 0-representation is excluded from the list of irreducible representations.

(4) *If C is a set of nonzero elements of a vector space V, then in the strong direct sum $\sum_{r=1}^{\infty} \otimes^r V$ the vectors $\sum \otimes^r c$ $(c \in C)$ are linearly independent.*

2. **Proofs.** If the conclusion of (4) does not hold, there is a minimal nonempty finite subset D of C such that there are nonzero scalars $k(d)(d \in D)$ for which

$$(*) \qquad \sum_{d \in D} k(d) \otimes^r d = 0 \qquad (r = 1, 2, \cdots).$$

Since D clearly has at least two elements, there is a linear function v^* on V which is not constant on D. Replacing r by $r+s$ in (*), taking the tensor product with $\otimes^s v^*$, and then contracting, we get

$$\sum_{d \in D} (k(d) \otimes^r d) v^*(d)^s = 0 \qquad (r = 1, 2, \cdots; s = 0, 1, 2, \cdots).$$

Thus if $k_1, k_2, \cdots, k_n$ are the distinct values taken by v^* on D, the value k_1 being taken on the subset D_1 of D, then because the van der Monde matrix (k_t^s) $(1 \leqq t \leqq n, 0 \leqq s \leqq n-1)$ is nonsingular, the equations (*) hold with D replaced by D_1, contradicting the minimal nature of D. Thus (4) is established.

Under the assumptions of (2) let $a = \sum k(b)b$ $(b \in B, k(b) \in K)$ be an element of A such that $(\otimes^r R)(a) = 0$ for $r = 1, 2, \cdots$. Then $\sum k(b) \otimes^r R(b) = 0$ for $r = 1, 2, \cdots$, each $k(b)$ is 0 by (4), whence a is also 0. Thus (2) is proved.

For the proof of (3) one may assume that $\{{}^rR\}$ is finite and consists of finite-dimensional representations. Let ${}^rM = {}^rM_0 \supset {}^rM_1 \supset \cdots$ be a composition series for the A-module ${}^rM$ corresponding to ${}^rR$, and let N be an arbitrary irreducible A-module. If A^0 is the radical of A, then A/A^0 is a sum of minimal left ideals. Hence there is a minimal left ideal I/A^0 such that $IN \neq 0$, and then there is a corresponding pair (r, i) such that $I({}^rM_i/{}^rM_{i+1}) \neq 0$, since otherwise I would be nilpotent because $\{{}^rR\}$ is complete and thus would be contained in A^0. If m and n are nonzero elements of ${}^rM_i/{}^rM_{i+1}$ and N respectively, it is then readily verified that the map $im \to in$ $(i \in I)$ is an A-module isomorphism of ${}^rM_i/{}^rM_{i+1}$ on N. Hence (3).

Reference

1. W. Burnside, *Theory of groups of finite order*, 2nd ed., Cambridge Univ. Press, Cambridge, 1911.

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