THE EXISTENCE OF COMPACT LINEAR MAPS BETWEEN BANACH SPACES

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In [2] J. D. Weston proves that given any separable Banach space Y, there exist a normed linear space X and a compact one-one linear operator which maps the conjugate space X' onto a subspace dense in Y.

It is the purpose of this paper to solve the following problem.

Given normed linear space X and Banach space Y, under what conditions does there exist a one-one compact linear map from X onto a subspace dense in Y?

Necessary and sufficient conditions for such an operator to exist are given (cf. (C) of the theorem below).

We first introduce some notations.

Suppose X and Y are normed linear spaces. Then $\mathcal{B}(X, Y)$ (resp., $\mathcal{K}(X, Y)$) is the space of all bounded (resp., compact) linear maps from X to Y. $\mathcal{B}_o(X, Y)$ (resp., $\mathcal{K}_o(X, Y)$) is the set of all one-one maps in $\mathcal{B}(X, Y)$ (resp., $\mathcal{K}(X, Y)$). $\mathcal{B}_d(X, Y)$ (resp., $\mathcal{K}_d(X, Y)$) is the set of all maps in $\mathcal{B}(X, Y)$ (resp., $\mathcal{K}(X, Y)$) with range dense in Y. $\mathcal{B}_{o,d}(X, Y) = \mathcal{B}_o(X, Y) \cap \mathcal{B}_d(X, Y)$, and $\mathcal{K}_{o,d}(X, Y) = \mathcal{K}_o(X, Y) \cap \mathcal{K}_d(X, Y)$. Finally, $\varnothing$ is the void set.

If X is a normed linear space and A is a subset of X', then A is *total* if and only if for each $x \neq 0$ in X there exists an x' in A such that $x'x \neq 0$. The following preliminary remarks are easily verified. The first of these gives alternative ways of stating a condition arising prominently in the rest of the paper.

(i) If X is a normed linear space, then X' contains a countable total subset if and only if X' is separable with respect to the w^* topology and also if and only if X' contains a total separable linear subspace.

(ii) If X is a separable normed linear space, then each of the conjugate spaces X' and X'' contains a countable total subset.

(iii) If X and Y are normed linear spaces, $\mathcal{B}_o(X, Y) \neq \varnothing$, and Y' has a countable total subset, then X' has a countable total subset.

DEFINITION. Suppose X is a Banach space, and suppose x_k is in X and ϵ_k is a real number for $k=1, 2, \cdots$. Then $\{x_k\}_1^\infty$ is $\{\epsilon_k\}_1^\infty$-*independent* if and only if

(a) $\sum_{k=1}^\infty \|\epsilon_k x_k\| < \infty$

and

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(b) for each bounded sequence $\{\alpha_k\}_1^\infty$ of scalars, $\sum_{k=1}^\infty \alpha_k\epsilon_k x_k=0$ implies $\alpha_k=0$ $(k=1, 2, \cdots)$.

REMARK. $\epsilon_k\neq 0$ $(k=1, 2, \cdots)$ if $\{x_k\}_1^\infty$ is $\{\epsilon_k\}_1^\infty$-independent.

LEMMA. *Suppose $\{x_k\}_1^\infty$ is a linearly independent sequence of elements in a Banach space X. Then there exists a sequence $\{\epsilon_k\}_1^\infty$ of positive real numbers such that $\{x_k\}_1^\infty$ is $\{\epsilon_k\}_1^\infty$-independent.*

PROOF. By considering the sequence $\{x_k/\|x_k\|\}_1^\infty$, we may assume $\|x_k\|=1$ $(k=1, 2, \cdots)$.

For each positive integer n, let $l(n)$ be the Banach space of n-tuples of scalars with norm defined by $\|(\eta_1, \eta_2, \cdots, \eta_n)\|=\sum_{k=1}^n |\eta_k|$. The map f defined by $f(\eta_1, \eta_2, \cdots, \eta_n)=\|\sum_{k=1}^n \eta_k x_k\|$ is a continuous map from $l(n)$ into the space of real numbers. Define S_n to be the set of all $(\eta_1, \eta_2, \cdots, \eta_n)$ in $l(n)$ such that $1/2\leqq\|(\eta_1, \eta_2, \cdots, \eta_n)\|\leqq n$. Then S_n is a compact subset of $l(n)$. Hence f attains a minimum a_n on S_n. Since $\{x_k\}_1^\infty$ is linearly independent, $a_n>0$. Clearly, $1>a_n\geqq a_{n+1}$ $(n=1, 2, \cdots)$.

Let $\epsilon_1=1$. Define $\epsilon_{n+1}=2^{-3}\epsilon_n a_n$ for $n=1, 2, \cdots$. Now

$$0<\epsilon_{n+j}\leqq(2^{-3})^j\epsilon_n a_n\leqq 2^{-3j}\quad (j, n=1, 2, \cdots). \tag{1}$$

Consider a bounded sequence $\{\alpha_k\}_1^\infty$ of scalars not all 0. Let $\alpha=\sup_{k=1}^\infty|\alpha_k|$. Then $\alpha>0$, and there exists an integer N such that $|\alpha_N|>\alpha/2$. Suppose $|\epsilon_{n_0}\alpha_{n_0}|=\max_{j=1}^N|\epsilon_j\alpha_j|$. Then $|\epsilon_{n_0}\alpha_{n_0}|\geqq|\epsilon_N\alpha_N|>\epsilon_N\alpha/2>0$, and

$$\left\|\sum_{j=1}^N \epsilon_j\alpha_j x_j\right\| = |\epsilon_{n_0}\alpha_{n_0}|\left\|\sum_{j=1}^N \epsilon_j\alpha_j x_j/(\epsilon_{n_0}\alpha_{n_0})\right\| \geqq |\epsilon_{n_0}\alpha_{n_0}|\, a_N > \alpha\epsilon_N a_N/2. \tag{2}$$

From (1) we have

$$\left\|\sum_{j=1}^\infty \epsilon_{N+j}\alpha_{N+j}x_{N+j}\right\| \leqq \alpha\sum_{j=1}^\infty \epsilon_{N+j} \leqq \alpha\sum_{j=1}^\infty 2^{-3j}\epsilon_N a_N < \alpha\epsilon_N a_N/4. \tag{3}$$

By (2) and (3),

$$\left\|\sum_{j=1}^\infty \epsilon_j\alpha_j x_j\right\| \geqq \left\|\sum_{j=1}^N \epsilon_j\alpha_j x_j\right\| - \left\|\sum_{j=1}^\infty \epsilon_{N+j}\alpha_{N+j}x_{N+j}\right\|$$
$$> (\alpha\epsilon_N a_N/2) - (\alpha\epsilon_N a_N/4) = \alpha\epsilon_N a_N/4 > 0.$$

Thus (b) is proved. That (a) holds follows from the fact that

$$0<\sum_{k=1}^\infty \epsilon_k \leqq \sum_{j=0}^\infty 2^{-3j} < \infty.$$

THEOREM. *Suppose X is an infinite-dimensional normed linear space and Y is an infinite-dimensional Banach space. Then* (A)–(C) *below hold.*

(A) $\mathcal{K}_o(X, Y) \neq \emptyset$ *if and only if X' has a denumerable total subset.*

(B) $\mathcal{K}_d(X, Y) \neq \emptyset$ *if and only if Y is separable.*

(C) $\mathcal{K}_{o,d}(X, Y) \neq \emptyset$ *if and only if* $\mathcal{K}_o(X, Y) \neq \emptyset$ (*alternatively,* $\mathcal{B}_o(X, Y) \neq \emptyset$) *and* $\mathcal{K}_d(X, Y) \neq \emptyset$, *i.e., if and only if Y is separable and X' has a denumerable total subset.*

Moreover, each of the sets $\mathcal{K}_o(X, Y)$, $\mathcal{K}_d(X, Y)$, $\mathcal{K}_{o,d}(X, Y)$ *which is nonvoid contains a map which is the limit* (*in norm*) *of continuous linear maps having finite-dimensional range.*

PROOF. There exist linearly independent sequences $\{x_k'\}_1^\infty$ and $\{y_k\}_1^\infty$ in X' and Y respectively such that $\|x_k'\| = \|y_k\| = 1$ $(k = 1, 2, \cdots)$. In addition, we may take $\{x_k'\}_1^\infty$ total in X' if X' has a denumerable total subset, and we may take $\{y_k\}_1^\infty$ spanning a dense subset of Y if Y is separable. By the lemma, $\{x_k'\}_1^\infty$ is $\{\epsilon_k\}_1^\infty$-independent and $\{y_k\}_1^\infty$ is $\{\eta_k\}_1^\infty$-independent for some $\{\epsilon_k\}_1^\infty$ and $\{\eta_k\}_1^\infty$. Let $T: X \to Y$ and $T_n: X \to Y$ be defined by $Tx = \sum_{k=1}^\infty \epsilon_k \eta_k x_k'(x) y_k$ and $T_n x = \sum_{k=1}^n \epsilon_k \eta_k x_k'(x) y_k$. Clearly, T_n is a bounded linear operator with finite-dimensional range and hence is compact. Moreover, $\{T_n\}_1^\infty$ converges in norm to T, for if x is in X and $\|x\| = 1$, then $\|T_n x - Tx\| \leq \sum_{k=n+1}^\infty \epsilon_k \eta_k \to 0$. Therefore T is a compact linear operator.

Suppose X' has a denumerable total subset. Consider x in X such that $Tx = 0$. Since $\{y_k\}_1^\infty$ is $\{\eta_k\}_1^\infty$-independent, $\epsilon_k x_k' x = 0$ and hence $x_k' x = 0$ $(k = 1, 2, \cdots)$. Therefore $x = 0$ since $\{x_k'\}_1^\infty$ is total in X'. Thus T is one-one, T is in $\mathcal{K}_o(X, Y)$, and $\mathcal{K}_o(X, Y) \neq \emptyset$.

Suppose Y is separable. To show that $(TX)^- = Y$, suppose the contrary. Then there exists $y' \neq 0$ in Y' such that $0 = y'TX$, whence for each x in X,

$$0 = y'\left(\sum_{k=1}^\infty \epsilon_k \eta_k x_k'(x) y_k\right) = \sum_{k=1}^\infty \epsilon_k \eta_k y'(y_k) x_k' x = \left(\sum_{k=1}^\infty \epsilon_k \eta_k y'(y_k) x_k'\right)(x).$$

Therefore $\sum_{k=1}^\infty \epsilon_k \eta_k y'(y_k) x_k' = 0$. Since $\{x_k'\}_1^\infty$ is $\{\epsilon_k\}_1^\infty$-independent, it follows that $y' y_k = 0$ $(k = 1, 2, \cdots)$. Hence $y' = 0$ since $\{y_k\}_1^\infty$ generates a subspace dense in Y. Thus we have contradicted the statement that $y' \neq 0$. Since $(TX)^- = Y$, T is in $\mathcal{K}_d(X, Y)$, and $\mathcal{K}_d(X, Y) \neq \emptyset$.

If X' has a denumerable total subset and Y is separable, then T is in $\mathcal{K}_o(X, Y) \cap \mathcal{K}_d(X, Y) = \mathcal{K}_{o,d}(X, Y)$, and $\mathcal{K}_{o,d}(X, Y) \neq \emptyset$.

Half of each of (A)–(C) has been proved. The other half of (A) follows from (iii) with the use of (ii) and the observation that for each T in $\mathcal{K}_o(X, Y)$, $T(A)$ is separable and T is in $\mathcal{B}_o(X, T(A))$. The other half of (B) follows from the fact that the range of each map in $\mathcal{K}(X, Y)$ is separable. The other half of (C) follows from (A) and (B). The last statement of the theorem is now clear from the proof thus far.

COROLLARY. *Suppose X and Y are separable infinite-dimensional normed linear spaces with Y a Banach space. Then* $\mathcal{K}_{o,d}(X, Y)$ *and* $\mathcal{K}_{o,d}(X', Y)$ *are nonempty.*

PROOF. Apply (ii) and the theorem.

The proof of the theorem leads to the following observations.

OBSERVATIONS. *Suppose X is an infinite-dimensional normed linear space and Y is an infinite-dimensional separable Banach space. Then* (A')–(C') *below hold.*

(A') $\mathcal{B}_o(X, Y) \neq \emptyset$ *if and only if X' has a denumerable total subset.*

(B') $\mathcal{B}_d(X, Y) \neq \emptyset$.

(C') $\mathcal{B}_{o,d}(X, Y) \neq \emptyset$ *if and only if* $\mathcal{B}_o(X, Y) \neq \emptyset$, *i.e., if and only if X' has a denumerable total subset.*

One could remove "infinite-dimensional" from the hypotheses of the theorem and the observations and still obtain "if and only if" results similar to, but slightly more complicated than, (A)–(C) and (A')–(C'). The details are completely straightforward.

Statement (A') will now be used to generalize a theorem of J. A. Clarkson.

A norm $\| \ \|$ on a linear space X is called *strictly convex* if and only if $\|x+y\| < \|x\| + \|y\|$ for all x and y in X which are linearly independent. Suppose X and Y are normed linear spaces with the norm on Y strictly convex, and suppose T is in $\mathcal{B}_o(X, Y)$. The norm $\| \ \|'$ on X such that $\|x\|' = \|x\| + \|Tx\|$ for each x in X is strictly convex and is equivalent to the original norm on X. Hence, applying (A') with Y a Hilbert space, one obtains immediately the following result, whose special case resulting from taking X separable is due to J. A. Clarkson [1, Theorem 9].

PROPOSITION. *Suppose X is a normed linear space whose conjugate space has a countable total subset. (For example, suppose X is a separable normed linear space or the conjugate space of a separable normed linear space.) Then the norm of X is equivalent to a strictly convex norm.*

REFERENCES

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