ON L_n SETS, THE HAUSDORFF METRIC, AND CONNECTEDNESS

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1. **Introduction.** A set S in the Euclidean plane E_2 is called an L_n set provided any two points in S can be joined by a polygonal line, of at most n segments, lying entirely in S. Horn and Valentine [1] characterized such sets for the case $n=2$ and studied properties of the complements of such sets. It is clear that the notion of L_n sets can be regarded as a generalization of the notion of convex sets and as a specialization of the notion of connected sets. Since polygonally connected sets are in some sense a limiting case of L_n sets, it seems reasonable to expect that L_n sets can be used to approximate polygonally connected sets. As we shall see in the sequel, any compact connected set (whether polygonally connected or not) can be approximated by compact L_n sets. We will use the notation $\langle p_1, p_2, \cdots, p_{n+1}\rangle$ for the n-sided polygonal line (n-line) joining p_1 to p_{n+1} where $p_2, \cdots, p_n$ are consecutive, intermediate vertices.

2. **The metric space $\mathcal{K}$.** In what follows, we will make use of a well-known metric space whose elements are the compact sets of the plane.

Let S be a compact set in E_2 and let $\epsilon>0$. The set

$$S_{(\epsilon)} \equiv \{p \in E_2: \rho(p, S) \leqq \epsilon\}$$

is called the ϵ-parallel body of S. Here ρ is the Euclidean metric of E_2.

THEOREM 1. *The parallel body $S_{(\epsilon)}$ of an L_n set S is an L_n set for the same n.*

PROOF. Let x and y be points of $S_{(\epsilon)}$. There exist points x' and y' in S for which $\rho(x, x') \leqq \epsilon$ and $\rho(y, y') \leqq \epsilon$. Now x' and y' can be joined by an n-line $\langle x', p_2, \cdots, p_n, y'\rangle$ lying in S. The segments $\langle x, p_2\rangle$ and $\langle p_n, y\rangle$ lie in the parallel body $\{\langle x', p_2, \cdots, p_n, y'\rangle\}_{(\epsilon)}$ which in turn is contained in $S_{(\epsilon)}$. Therefore, $\langle x, p_2, \cdots, p_n, y\rangle$ is an n-line in $S_{(\epsilon)}$ joining x and y.

Let S and T be compact sets in E_2. Let,

$$d(S, T) = \inf\{\epsilon: T \subset S_{(\epsilon)} \text{ and } S \subset T_{(\epsilon)}\}.$$

Then d is a metric on $\mathcal{K}$ the class of all compact sets in E_2. For a discussion of some of the properties of $\mathcal{K}$ see [2]. In particular, it is shown there that $\mathcal{K}$ is complete. Some additional properties of $\mathcal{K}$ are enumerated in

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THEOREM 2. (a) *If* $S \in \mathcal{K}$ *then* $S = \lim_{\epsilon \to 0} S_{(\epsilon)}$.

(b) *If* $S^1, S^2, \cdots$ *is a decreasing sequence of sets, each a member of* $\mathcal{K}$, *then*

$$\lim_{k \to \infty} S^k = \bigcap_{k=1}^{\infty} S^k.$$

(c) *Let* n *be a positive integer. The limit* S *of a sequence* $\{S^k\}$ *of compact* L_n *sets is a compact* L_n *set.*

PROOF. The proof of part (a) is obvious and the proof of part (b) is contained in the proof of completeness of $\mathcal{K}$ referred to above. We turn therefore to part (c). Let $d(S^k, S) = \epsilon_k$ and let p and q be points of S. Then for each k, p and q are in $S^k_{(\epsilon_k)}$. By Theorem 1, $S^k_{(\epsilon_k)}$ is an L_n set. Thus there exist points $p_1^k, p_2^k, \cdots, p_{n-1}^k$ in $S^k_{(\epsilon_k)}$ such that the n-line $\langle p, p_1^k, p_2^k, \cdots, p_{n-1}^k, q\rangle$ is contained in $S^k_{(\epsilon_k)}$. Since $S^k_{(\epsilon_k)} \subset S_{(2\epsilon_k)}$ we have $\langle p, p_1^k, \cdots, p_{n-1}^k, q\rangle \subset S_{(2\epsilon_k)}$. We choose a subsequence of $\{S^k\}$ for which the corresponding subsequences $\{p_1^k\}, \cdots, \{p_{n-1}^k\}$ converge to the points $p_1, \cdots, p_{n-1}$. Since S is closed and $\epsilon_k \to 0$, the n-line $\langle p, p_1, \cdots, p_{n-1}, q\rangle$ lies in S.

3. **L_n sets and connectedness.** We are now ready to characterize compact, connected sets in terms of compact L_n sets. We begin with three lemmas.

LEMMA 1. *If* S *is a compact, connected set and* $\epsilon > 0$ *then there exists a positive integer* n *such that* $S_{(\epsilon)}$ *is an* L_n *set.*

PROOF. The collection of open ϵ-discs about points of S forms a cover of S from which a finite subcover $D_1, \cdots, D_N$ can be selected. Consider the network formed by the line segments joining the centers of each pair D_i and D_j of overlapping discs. The centers of any two of the N discs are joined by an $(N-1)$-line of this network. It follows that any two points of S can be joined by an $(N+1)$-line in $\bigcup_{i=1}^N D_i$, hence any two points of $S_{(\epsilon)}$ can be joined by an $(N+3)$-line in $S_{(\epsilon)}$.

LEMMA 2. *The limit* S *of a sequence* $\{S^k\}$ *of compact, connected sets is compact and connected.*

PROOF. Using the fact that $\{S^k\}$ is a Cauchy sequence we can select a subsequence of $\{S^k\}$ and parallel bodies of the members of this subsequence, such that these parallel bodies form a decreasing sequence of sets approaching S. By Lemma 1, these parallel bodies are connected. The conclusion of the lemma follows from Theorem 2(b) and the fact that the intersection of a decreasing sequence of compact, connected sets is compact and connected.

LEMMA 3. *Any compact, connected set S is the limit of a sequence of compact L_n sets.*

The proof follows immediately from Lemma 1 and Theorem 2(a).

The following theorem is an immediate consequence of Lemma 2 and Lemma 3.

THEOREM 3. *A necessary and sufficient condition that the set S be compact and connected is that S be the limit of a sequence of compact L_n sets.*

REFERENCES

1. Alfred Horn and F. A. Valentine, *Some properties of L_n sets in the plane*, Duke Math. J. **16** (1949), 131–140.

2. A. M. Macbeath, *Compactness theorems*, Seminar on Convex Sets, 1949–1950, Chapter V, pp. 41–51, The Institute for Advanced Study.

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