

NON-AMPHICHEIRALITY OF THE SPECIAL ALTERNATING LINKS

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Introduction. An oriented (polygonal) link² k is said to be *amphicheiral* if it is isotopic to its mirror imaged link. Our study started in the direction of proving the following conjecture.

CONJECTURE. *All special alternating³ links are not amphicheiral.*

As is well known, all torus knots are not amphicheiral. Any torus link that is alternating and prime is an elementary torus link, i.e. a torus link of type $(n, 2)$, and any prime special alternating link for which $\Delta(0) = 1$ is an elementary torus link, where $\Delta(t)$ denotes the A -polynomial⁴ of a link. In this paper, the following will be proved.

THEOREM A. *A special alternating link for which $\Delta(0) = 2$ is not amphicheiral.*

THEOREM B. *A special alternating knot of odd genus is not amphicheiral.*

1. Let κ be a link type and let k be an oriented link representing κ . Let K be a (*normed regular*) projection of k onto $S^2 \subset S^3$ [8]. K is oriented by the orientation induced by that of k . Let $N(K)$ denote the number of crossing points in K and let $N(\kappa)$ denote the minimum of the crossing points in projections of any link representing κ . $N(\kappa)$ is a well-known link invariant. A projection K is said to be *reduced* if $N(K) = N(\kappa)$. Then we have the following classical theorem [2; 4].

THEOREM 1.1. *If a reduced projection of k is special alternating, then k is not amphicheiral.*

Although every link possesses at least one special projection, there exist alternating links that have no special alternating projection [5]. In the following by the *projection* of an alternating, or special alternating, link will be meant an *alternating*, or *special alternating*, projection.

Since the product of the second torsion numbers of a knot is equal

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² A link of multiplicity $\mu (\geq 1)$ is the union of μ ordered, pairwise disjoint simple closed curves in 3-sphere S^3 . A knot is a link of multiplicity one.

³ For the definition, see [5, p. 278].

⁴ For the definition, see [5, p. 280 and (1.11)]. We may assume without loss of generality that the leading coefficient of $\Delta(t)$, i.e. $\Delta(0)$, is greater than zero.

to $|\Delta(-1)|$, Goeritz's Theorem [8, p. 30] simply implies the following

THEOREM 1.2. *If $|\Delta(-1)| \equiv 3 \pmod{4}$ then the knot is not amphicheiral.*

2. Theorem B will be shown from Theorem 1.2 and the following

LEMMA 2.1. *If k is a special alternating knot of genus m , then $|\Delta(-1)| \equiv 2m+1 \pmod{4}$.*

PROOF. The A -polynomial of k is of the form [1; 5]:

$$\Delta(t) = a_0 - a_1t + a_2t^2 + \cdots + (-1)^i a_i t^i + \cdots + a_{2m}t^{2m},$$

where $a_0, a_1, \dots, a_{2m} > 0$. Then we have $\Delta(-1) = \sum_{i=0}^{2m} a_i > 0$. Since $a_j = a_{2m-j}$, for $j = 0, 1, \dots, m$, it follows that $\Delta(-1) = 2(a_0 + a_1 + \cdots + a_{m-1}) + a_m$. On the other hand, since $\Delta(1) = \sum_{i=0}^{2m} (-1)^i a_i = 1$ [5], we have

$$\begin{aligned} \Delta(-1) &= 2(a_0 + a_1 + \cdots + a_{m-1}) \\ &\quad + (-1)^m \{1 - 2[a_0 - a_1 + \cdots + (-1)^{m-1} a_{m-1}]\} \\ &= 2\{[1 + (-1)^{m+1}]a_0 + [1 + (-1)^{m+2}]a_1 + \cdots \\ &\quad + [1 + (-1)^{2m}]a_{m-1}\} + (-1)^m \\ &\equiv (-1)^m \equiv 1 + 2m \pmod{4}. \end{aligned}$$

3. Let G be a (*finite planar*), *directed* or *undirected*, *graph*. A sequence $P = (a_0, A_1, a_1, \dots, A_n, a_n)$, a_i and A_j denoting vertices and edges, is an *undirected path* in G from a_0 to a_n if a_0, a_1, \dots, a_{n-1} are all distinct and if a_{j-1} and a_j are two ends in G of A_j for $j = 1, 2, \dots, n$. Specially a path P is said to be *directed* from a_0 to a_n if A_j are directed from a_{j-1} to a_j . A path is called a *circuit* (or n -*circuit*) if $a_0 = a_n$. The *valency* of a vertex is defined as the number of edges in G incident to it.

Now let K be a projection of a link k . Then we can uniquely define a graph $G(K)$ and a dual graph $G^*(K)$ of K [5]. To each crossing in K there correspond a unique edge of $G(K)$ and a unique edge of $G^*(K)$. Each edge of $G(K)$ is directed in such a way that it crosses the overpassing segment from left to right at the crossing point corresponding to it. Thus $G(K)$ and $G^*(K)$ are directed. If k is a special alternating link, then we can choose the graph $G(K)$ in such a way that the valency of each vertex of $G(K)$ is even and $G(K)$ is the union of directed circuits.

Conversely given an undirected graph G , there are exactly two *alternating projections*. One is the projection of the mirror imaged

link of the other. If they are oriented, then G is directed. Since the A -polynomial of a mirror imaged link coincides with that of an original link, the A -polynomial of the alternating link induced from a given graph G is uniquely determined. Thus we can call it the A -*polynomial* of G .

We can assume, hereafter, that $G(K)$ and $G^*(K)$ are connected and contain no 1-circuit.

A projection K is of the *bridge form of type* (p_1, p_2, \dots, p_r) if $G(K)$ or $G^*(K)$ consists of r paths P_1, P_2, \dots, P_r from a vertex u to another vertex v , having no common vertices except u and v , where P_i contain p_i vertices other than u and v . k is called, then, a *bridge link of type* (p_1, p_2, \dots, p_r) . If k is a special alternating bridge link of type (p_1, p_2, \dots, p_r) , then in $G(K)$, r is always even, and one half of r paths are directed from u to v and the other half are directed reversely. The number of vertices of $G(K)$ equals⁵ $d(K)+1$. For example, the special alternating bridge link of type (p, q) coincides with the torus link of type $(p+q+2, 2)$. Moreover, a special alternating knot of genus one is an alternating bridge knot of type (p, q, r) .

4. To characterize the links for which $\Delta(0)=2$, we need the following

LEMMA 4.1. *Let G be the directed graph of a projection K of a special alternating link and let G' be the directed graph obtained from G by removing a directed circuit C . Denoting the A -polynomials of G and G' by $\Delta(t)$ and $\Delta'(t)$ respectively, we have*

$$(4.1) \quad \Delta(0) \geq \Delta'(0) + 2^{n-1} - 1,$$

where n denotes the number of vertices in $C \cap G'$.

PROOF. Let us denote $I_{p,q}(i, j)$ the $p \times q$ matrix whose elements are all zero except the (i, j) element, which is equal to one. By definition, then, knot matrices⁶ M, M' of G, G' are of the forms:

$$M = \begin{pmatrix} A_{00} & A_{01} & \cdots & A_{0,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n+1,0} & A_{n+1,1} & \cdots & A_{n+1,n+1} \end{pmatrix}, \quad M' = \begin{pmatrix} A & A_{0,n+1} \\ A_{n+1,0} & A_{n+1,n+1} \end{pmatrix},$$

where $A_{0i} = -I_{n,p_i}(i, 1)$, $A_{j,0} = -I_{p_j,n}(p_j, j+1)$, $A_{n,0} = -I_{p_n,n}(p_n, 1)$,

⁵ $d(k)$ or $d(K)$ denotes the degree of the A -polynomial $\Delta(t)$ of k .

⁶ By the *knot matrix* of G is meant the knot matrix of the alternating link induced by G . For the definition and properties of a knot matrix, and for the matrix notations, see [5; 6].

$$A_{kk} = I - \sum_{i=1}^{p_k-1} I_{p_k, p_k}(i, i+1),$$

($i = 1, \dots, n, j = 1, \dots, n-1, k = 1, \dots, n, p_l \geq 0$ for all l),

$A = A_{00} - I$, I denoting the identity matrix, and $A_{ij} = 0$ ($i, j = 1, \dots, n+1, i \neq j$). A_{00} and A correspond to n common vertices in $C \cap G'$. Since $\Delta(0) = \det \tilde{M}(n, n)$ and $\Delta'(0) = \det \tilde{M}'(n, n)$ by definition, it follows easily that

$$\Delta(0) = \Delta'(0) + \sum_{\substack{1 \leq i_1 < i_2 < \dots < i_\lambda \leq n-1 \\ \lambda = 1, 2, \dots, n-1}} \det N_{i_1 i_2 \dots i_\lambda};$$

$N_{i_1 i_2 \dots i_\lambda}$ is the matrix obtained from $\tilde{M}'(n, n)$ by replacing the i_j -row ($1 \leq j \leq \lambda$) by $R_{i_j} = I_{1, n+s-1}(1, i_j) - I_{1, n+s-1}(1, i_j+1)$, where s denotes the number of columns of $A_{0, n+1}$ and we define $R_{n-1} = I_{1, n+s-1}(1, n-1)$. To prove (4.1) it is sufficient to show that $\det N_{i_1 i_2 \dots i_\lambda} > 0$. This is obvious from Lemma 2.4 (i) in [5], since $N_{i_1 \dots i_\lambda}$ is of strongly special type on the row.⁷ q.e.d.

If $n = 2$, the Lemma 4.1 is sharpened as follows.

LEMMA 4.2. $\Delta(0) = \Delta'(0) + \Delta''(0)$, where $\Delta''(t)$ denotes the A -polynomial of the graph obtained from G' by identifying two common vertices.

By using Lemma 4.1, we can complete the characterization. In fact, we have

LEMMA 4.3. A prime special alternating link k for which $\Delta(0) = 2$ is a special alternating bridge link of type (p, q, r, s) , where $p+q+r+s+1 = d(k)$, and conversely.

PROOF. Let K be a projection of k and let G' be the graph obtained in Lemma 4.1. Then (4.1) shows that $2 = \Delta(0) \geq \Delta'(0) + 2^{n-1} - 1$. This implies that $\Delta'(0) = 1$ and $n = 2$, since k is prime. Thus K is of the form as is required in the lemma. It is obvious that $p+q+r+s+1 = d(k)$. Converse is clear.

5. To prove Theorem A, some lemmas on $N(\kappa)$ will be required. In Lemmas 5.1–5.3, k need not be alternating.

LEMMA 5.1. For any nontrivial link k , $N(\kappa) \geq d(k) + 1$. Equality holds if and only if k is an elementary torus link.

This is obvious from [1; 5].

LEMMA 5.2. For any special projection K of any link k for which $\Delta(0) = 2$, $N(K) \geq d(k) + 3$.

⁷ For the definition, see [5, p. 289].

PROOF. From Lemma 5.1, we see that $G(K)$ contains at least $d(K) + 1$ edges. Since K is special, the valencies of all vertices must be even. If the valency of each vertex is two, then G is a circuit, since G is connected. Thus we have $\Delta(0) = 1$, which is a contradiction. If only one vertex is of valency four, then G consists of two circuits having one vertex in common, from which it follows that $\Delta(0) = 1$. Thus G contains at least two vertices of valency ≥ 4 . It follows that $N(K) \geq d(K) + 3$.

LEMMA 5.3. *For any nonspecial projection K of any link, $N(K) > d(K) + 3$.*

PROOF. We can define a matrix for any link [5; 6]. Then Corollary 1.40 in [5] can be extended as follows. Let $K = K_1 * K_2 * \dots * K_p$, ($p \geq 2$),⁸ and let $\Delta_i(t)$ denote the A -polynomials of K_i . Then we have

$$(5.1) \quad \Delta(0) = \prod_{i=1}^p \Delta_i(0) \quad \text{and} \quad d(K) = \sum_{i=1}^p d(K_i).$$

Thus all $\Delta_i(0)$ except one, say $\Delta_p(0)$, are equal to one, and $\Delta_p(0) = 2$. Hence from Lemmas 5.1, 5.2 we have

$$(5.2) \quad N(K_i) \geq d(K_i) + 1 \quad \text{and} \quad N(K_p) \geq d(K_p) + 3.$$

Consequently it follows from (5.1) and Lemma 3.29 in [5],

$$\begin{aligned} N(K) &= \sum_{i=1}^{p-1} (d(K_i) + 1) + d(K_p) + 3 \\ &= \sum_{i=1}^p d(K_i) + (p - 1) + 3 \\ &= d(K) + p + 2 > d(K) + 3, \end{aligned} \quad \text{q.e.d.}$$

Now any special alternating bridge link k of type (p, q, r, s) possesses a projection K with $d(K) + 3$ crossing points, which is reduced and special alternating. Thus from Theorem 1.1 it follows that k is not amphicheiral. Combining with Lemma 4.2, we have Theorem A.

APPENDIX. Since a special alternating bridge link of type $(0, 0, p, q)$ ($p, q \geq 0$) is of two bridges, we can prove the non-amphicheirality of links of this kind by means of a theorem of H. Schubert [9]. In fact, we see that $\alpha = 2pq + 3p + 3q + 4$ and $\beta = 2p + 3$ or $2q + 3$, where (α, β) denotes the normal form of this link [7; 9]. From this it is easy to show that $\beta^2 \not\equiv -1 \pmod{2\alpha}$.

⁸ For the definition, see [5, p. 293].

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