ON A COMMUTATIVE EXTENSION OF A BANACH ALGEBRA[1]

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Introduction. Let A be a commutative semi-simple Banach algebra and let $\Delta(A)$ be the set of nonzero multiplicative functionals on A. Denote by A' the strongly closed span of $\Delta(A)$ and by A'' the Banach space adjoint of A'. Modifying a construction of R. Arens [1] we introduce a multiplication in A'' under which A'' becomes a *commutative* Banach algebra. A is algebraically isomorphic to a subalgebra of A'' and the isomorphism is continuous. Indeed, we will henceforth need that *the embedding of A in A'' be topological.* If A has a weak bounded approximate identity, then the algebra A^m of multipliers of A (see [3; 4]) is likewise embeddable in A'' and the isomorphism is again continuous. In this paper we are concerned with identification of A^m in A''. For example, if A is, in addition to the above assumptions, regular and Tauberian and if $\Delta(A)$ is discrete, then A^m and A'' are topologically and algebraically isomorphic. The main result is the following: For A with approximate identity in $j_A(\infty)$, an element F of A'' is a multiplier of A if and only if F belongs locally to $\hat{A}$ at each point of $\Delta(A)$.

The multiplier algebra of the group algebra $L_1(G)$ of a locally compact abelian group G is the algebra $M(G)$ of bounded measures on G. In this case, our main theorem closely parallels Eberlein's necessary and sufficient condition for a function to be a Fourier-Stieltjes transform of a measure on G. See [5]. As an application, we construct A'' and use Eberlein's theorem to determine the algebra of multipliers of the L_1-algebra of certain semi-groups G_+.

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1. **Preliminaries.** $\hat{A}$ is the function algebra on $\Delta(A)$ isomorphic to the commutative semi-simple Banach algebra A in the Gelfand theory. Let A^m be the set of all complex-valued functions on $\Delta(A)$ such that $f\cdot\hat{x}\in\hat{A}$ for all $x\in A$. Each $f\in A^m$ determines a bounded operator f on A given by $(fx)\hat{} = f\hat{x}$. The set of all such operators with the uniform operator norm is a commutative semi-simple Banach algebra under the obvious operations. It is called the algebra of multipliers

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of A and is designated by A^m. (The ambiguous use of A^m will be clear in context.) $\Delta(A)$ is homeomorphic to a subset of $\Delta(A^m)$ both in their weak* topologies. See [3].

A is said to be regular if functions in $\hat{A}$ separate points and closed sets of $\Delta(A)$. $j_A(\infty)$ is the set of $x \in A$ such that $\hat{x}$ has compact support. Regular A is called Tauberian, if $j_A(\infty)$ is dense in A.

A^* will denote the Banach space adjoint of A and A^{**}, the second adjoint. If f is a function on X and $Y \subset X$, $f|Y$ will denote the restriction of f to Y. Let K be the complex number field.

2. **Construction of A''.** For completeness, we first sketch the construction of R. Arens [1] used to introduce a multiplication in A^{**}.

Define fx ($x \in A$, $f \in A^*$) by $fx(y) = f(xy)$ for $y \in A$, define Ff ($f \in A^*$, $F \in A^{**}$) by $Ff(x) = F(fx)$ for $x \in A$, define GF (F, $G \in A^{**}$) by $GF(f) = G(Ff)$ for $f \in A^*$. fx is linear on A and $|fx(y)| = |f(xy)| \leqq \|f\| \|xy\| \leqq \|f\| \|x\| \|y\|$, hence $fx \in A^*$ and $\|fx\| \leqq \|f\| \|x\|$. Similarly, Ff is linear on A and $|Ff(x)| = |F(fx)| \leqq \|F\| \|fx\| \leqq \|F\| \|f\| \|x\|$, so $Ff \in A^*$ and $\|Ff\| \leqq \|F\| \|f\|$. Also GF is linear on A^*, $|GF(f)| = |G(Ff)| \leqq \|G\| \|Ff\| \leqq \|G\| \|F\| \|f\|$, therefore, $GF \in A^{**}$ and $\|GF\| \leqq \|G\| \|F\|$. As Arens points out, multiplication so defined in A^{**} is not in general commutative.

In the above construction replace A^* by the strongly closed span A' of $\Delta(A)$. Indeed, we might even replace A^* by the strongly closed span of the Silov boundary $\partial\Delta(A)$ of $\Delta(A)$. ($\partial\Delta(A)$ is the minimal closed set on which each function $|\hat{x}| \in |\hat{A}|$ maximizes.) We then introduce as in the preceding paragraph a multiplication in $A'' = A'^*$ which is defined on a dense subset of A' as follows (all sums are finite):

(i) $(\sum a_\nu p_\nu)x = \sum a_\nu p_\nu(x) p_\nu$ $\qquad (p_\nu \in \Delta(A),\ x \in A,\ a_\nu \in K)$,

(ii) $F(\sum a_\nu p_\nu) = \sum a_\nu F(p_\nu) p_\nu$ $\qquad (F \in A'',\ \sum a_\nu p_\nu \in A')$,

(iii) FG is given by $FG(\sum a_\nu p_\nu) = \sum a_\nu F(p_\nu) G(p_\nu)$
$\qquad (F, G \in A'', \sum a_\nu p_\nu \in A')$

FG clearly extends to a function on all of A'. Multiplication so defined is commutative and exactly as in the Arens construction $\|FG\| \leqq \|F\| \|G\|$.

3. **Embeddings.** In a natural fashion A can be embedded in A'' and the embedding is continuous. Let x^{**} be the element of A^{**} associated, under the canonical embedding of A in A^{**}, with $x \in A$. Then $x^{**}|A' \in A''$. And

$$\langle \sum a_\nu p_\nu, (x^{**}|A')(y^{**}|A') \rangle = \sum a_\nu (x^{**}|A')(p_\nu)(y^{**}|A')(p_\nu)$$

which agrees with the multiplication in A'' given by (iii) above. Observe that $x^{**}|\Delta(A) = \hat{x}$.

In order to realize A^m as a subalgebra of A'', we assume that A has a weak bounded approximate identity; that is, there exists a net $\{x_\delta\}$ in A such that for each finite set $\{p_1, p_2, \cdots, p_n\} \subset \Delta(A)$, $\lim_\delta x_\delta(p_\nu) = 1$ for $\nu = 1, 2, 3, \cdots, n$ and $\{x_\delta\}$ is strongly bounded. Then

$$
\begin{aligned}
\left| F \sum a_\nu p_\nu \right| &= \left| \sum a_\nu F(p_\nu) \right| = \lim_\delta \left| \sum a_\nu F \hat{x}_\delta(p_\nu) \right| \\
&= \lim_\delta \left| F \hat{x}_\delta \sum a_\nu p_\nu \right| \leqq \lim_\delta \|F x_\delta\| \left\| \sum a_\nu p_\nu \right\| \\
&\leqq k \|F\| \left\| \sum a_\nu p_\nu \right\|
\end{aligned} \tag{1}
$$

for each $F \in A^m$ where $\|x_\delta\| \leqq k$ for all δ. (1) implies that A^m is embedded continuously in A''. Semi-simplicity of A^m assures that the embedding is one-one. Existence of an approximate identity in the strong sense—a bounded net $\{x_\delta\} \subset A$ such that $\|x_\delta x - x\| \to 0$ for all $x \in A$—evidently suffices to accomplish this embedding.

4. **Example.** We note the form of A'' when $A = L_1(G)$ with G a locally compact Abelian group: A' is the strongly closed span of $\hat{G}$, the dual of G. $\hat{G}$ is contained in the essential bounded functions $L_\infty(G)$ on G. So the norm in this case is the essential supremum norm of $L_\infty(G)$. It is well know that the uniform closure of the characters of G are precisely the almost periodic functions on G, which in turn can be identified as the continuous functions on the almost periodic compactification G^* of G. Denote this sup-normed Banach algebra by $C(G^*)$. The adjoint of $C(G^*)$ is $M(G^*)$. $M(G)$ is isometrically isomorphic to a subset of $M(G^*)$. The multipliers of $L_1(G)$ (viz. $M(G)$) are precisely the measures of $M(G^*)$ which have weak* continuous restrictions to $\hat{G}$. This last result is due to Eberlein [5]. Also see Glicksberg [6].

Theorem. *Let A be a regular Tauberian commutative semi-simple Banach algebra with discrete maximal regular ideal space Δ. If A is topologically embedded in A'' and A^m is embeddable in A'', then $A^m = A''$.*

Proof. Let $J = \{x \in A : F\hat{x} \in \hat{A}$ where $F \in A''\}$. Since A is regular and Δ is discrete,

$$I = \{x_p : \hat{x}_p(p) = 1, \hat{x}_p(p') = 0 \text{ for } p \neq p';\ p, p' \in \Delta\} \subset J.$$

Indeed, span $(I) \subset J$. Consequently, $F \cdot j_A(\infty)\hat{} \subset \hat{A}$. But $x_n \to x$ in A, $x_n \in j_A(\infty)$, implies $Fx \in A$, since $Fx_n \to Fx$ in A'' and A is embedded topologically in A''. So $F \cdot j_A(\infty)^{-\hat{}} \subset \hat{A}$. Therefore $A'' = A^m$.

Remark. However, even when A is regular and topologically embedded in A'', and when A^m is embeddable in A'', it is not true that A^m is the set of those elements of A'' whose restrictions to Δ are weak* continuous, as one might hope from the previous example and theorem. Specifically, if D is the algebra of complex-valued functions on $[0, 1]$ possessing continuous first derivatives, functionals in A'' can be identified as indefinite integrals of functions in $L_\infty[0, 1]$, whereas $D^m = \hat{D}$, the set of continuously differentiable functions on $[0, 1]$.

5. **Main theorem.** Under certain restrictions, however, we are able to obtain a necessary and sufficient condition that a functional in A'' actually be a multiplier of A.

DEFINITION. Let A be a commutative semi-simple Banach algebra with regular maximal ideal space Δ. A function f on Δ is said to belong locally to $\hat{A}$ at $p \in \Delta$ if there exists a neighborhood V of p and a function $\hat{x} \in \hat{A}$ such that $f| V = \hat{x}| V$. f belongs locally to $\hat{A}$ at ∞ if there exists a neighborhood W of ∞ in $\Delta \cup \{\infty\}$ and a function $\hat{x} \in \hat{A}$ such that $f| W = \hat{x}| W$.

In the lemmas which follow we assume that A is a regular commutative semi-simple Banach algebra, that A is embedded topologically in A'' and that A^m is embeddable in A''.

LEMMA 1. *If $f \in A^m$, then f belongs locally to $\hat{A}$ at each $p \in \Delta$.*

PROOF. Let $p \in \Delta$. Choose a compact neighborhood V and a function $\hat{x} \in \hat{A}$ such that $\hat{x}(p) = 1$ for $p \in V$ (possible, by regularity). Then, for $f \in A^m$, $f\hat{x} \in \hat{A}$ and $f\hat{x}| V = f| V$.

Our next lemma is essentially established in [7, p. 85].

LEMMA 2. *If a function f on Δ belongs locally to $\hat{A}$ at each $p \in \Delta \cup \{\infty\}$, then $f \in \hat{A}$.*

PROOF. If $x_i \in A$ and $f = \hat{x}_i$ on U_i $(i = 1, 2)$ and if C is a compact subset of U_2, then there exists $y \in A$ such that $f = \hat{y}$ on $U_1 \cup C$: Using regularity, let $e \in A$ be such that $\hat{e} = 1$ on C and $\hat{e} = 0$ on $\Delta - U_2$. Define $y = x_2 e + x_1(1 - e)$. $\hat{y} = \hat{x}_2 = f$ on C, $\hat{y} = \hat{x}_1 = f$ on $U_1 - U_2$, and $\hat{y} = f\hat{e} + f(1 - \hat{e}) = f$ on $U_1 \cap U_2$. Now cover $\Delta \cup \{\infty\}$ with finitely many open sets $U_1, U_2, \cdots, U_n$ such that $f = \hat{x}_i$ on U_i and $x_i \in A$ $(i = 1, 2, \cdots, n)$. We can find open sets $V_1, V_2, \cdots, V_n$ also covering $\Delta \cup \{\infty\}$ such that $V_i \subset \overline{V}_i \subset U_i$ $(i = 1, 2, \cdots, n)$. Using the first assertion of the proof and iterating, we have $f = \hat{y}$ on Δ with $y \in A$.

If $F \in A''$ and if for each $p \in \Delta \cup \{\infty\}$, there is a neighborhood V of p with $F\hat{A}| V \subset \hat{A}| V$, then $F \in A^m$, since $F\hat{x}$ $(x \in A)$ belongs locally to $\hat{A}$ at each $p \in \Delta \cup \{\infty\}$ and the previous lemma applies.

LEMMA 3. *If* $x \in (xj_A(\infty)^-)$ *for every* $x \in A$ *and if* $F \in A''$ *belongs locally to* $\hat{A}$ *at each* $p \in \Delta$, *then* $F \in A^m$.

PROOF. Choose $\{x_n\} \in j_A(\infty)$ with the property that $x_n x \to x$ in A. Clearly, $F\hat{x}_n$ belongs locally to $\hat{A}$ at each $p \in \Delta \cup \{\infty\}$, hence $F\hat{x}_n \in \hat{A}$. For any $\epsilon > 0$,

$$\|Fx_n - Fx\| \leqq \|F\| \|x_n x - x\| < \epsilon \qquad \text{for large } n.$$

And by the topological embedding of A in A'' we have that $Fx \in A$. Therefore $F \in A^m$.

The existence of an approximate identity $\{u_\delta\} \subset j_A(\infty)$ suffices for x to be in $xj_A(\infty)^-$ and for A^m to be embeddable in A''.

THEOREM. *Let* A *be a regular commutative semi-simple Banach algebra with an approximate identity* $\{u_\delta\} \subset j_A(\infty)$ *and such that* A *is topologically embeddable in* A''. *Then* $F \in A''$ *is a multiplier of* A *if and only if* F *belongs locally to* $\hat{A}$ *at each point of* Δ.

The striking similarity of the above theorem with Eberlein's characterization of Fourier-Stieltjes transforms of measures [5] can be emphasized by reformulating the theorem without specific reference to A''. That F be in A'' simply requires: there exist a constant k such that

$$\left| \sum a_\nu F(p_\nu) \right| \leqq k \left\| \sum a_\nu p_\nu \right\| \tag{2}$$

for all $p_1, p_2, \cdots, p_n \in \Delta$ and complex numbers $a_1, a_2, \cdots, a_n$. Eberlein's proof can be viewed as a verification of the fact that continuous functions on $\Delta = G^\wedge$ satisfying (2) belong locally to $\hat{A} = L_1(G)^\wedge$ and hence are in $A^m = M(G)$.

However, it seems worthwhile to give a direct proof[2] that (2) implies F belongs locally to $L_1(G)^\wedge$. By regularity, if D is a compact neighborhood of some point of $G^\wedge$ and E is an open subset containing D, then there exists an $f \in L_1(G)$ such that $\hat{f}(p) = 1$ for $p \in D$ and $\hat{f}(p) = 0$ for $p \in \hat{G} - E$. Consequently, $H = F \cdot \hat{f}$ is in $L_1 \cap L_2(G)$. Let T^* denote the inverse Fourier transform. To have F belong locally to $L_1(G)^\wedge$ it is sufficient to show that $T^*H \in L_2(G)$ is also in $L_1(G)$. Let U be an element in $C(\hat{G})$ of an approximate identity on $\hat{G}$ with $\hat{U}$ having compact support. Then by the Plancherel theorem,

$$\left| \int_G \sum a_\nu p_\nu(x) T^*H(x) \hat{U}(x) dx \right| = \left| \int_{G^\wedge} \left(\sum a_\nu L_{p_\nu} H \right)(p) U(p) dp \right| \tag{3}$$

[2] Added on May 25, 1962. This proof is due essentially to Professor I. Glicksberg, who provided corrections needed to circumvent certain errors in the author's original argument.

where L_{p_ν} denotes left translation by p_ν^{-1}. Furthermore,

$$\sup_{p\in\hat{G}} \left| \sum a_\nu L_{p_\nu} H(p) \right| \leq k \left\| \sum a_\nu p_\nu \right\|_\infty \tag{4}$$

by (2). Let the support of $\hat{U}$ be S. S is compact in the almost periodic compactification G^* of G, so by Tietze's theorem we can find an almost periodic function $h \in C(G^*)$ such that

$$h \mid S = \exp(-i \arg(T^*H) \cdot \hat{U})$$

with $\|h\|_\infty \leqq 1$. Then h can be uniformly approximated on G^* by a trigonometric polynomial $\sum a_\nu p_\nu$ of modulus less than $1+\epsilon$. Using this trigonometric polynomial in (3) and (4) we have

$$\int_G \left| (T^*H)\hat{U} \right| \leqq \left| \int_G (T^*H) \cdot \sum a_\nu p_\nu \cdot \hat{U} \right| \leqq M$$

where M is a constant. And since we can make $|\hat{U}| > 1/2$ on any compact subset C of G,

$$\int_C \left| T^*H \right| \leqq 2M$$

on an arbitrary compact subset C of G, thereby showing that T^*H is in $L_1(G)$.

6. **An application.** Let G_+ be a closed semi-group of G such that the interior of G_+ is dense in G_+, $1 \in G_+$, and the interior of G_+ generates G. For such semi-groups G_+ we propose to identify the multipliers of $L_1(G_+) = \{f \in L_1(G) : f \text{ vanishes off } G_+\}$. $L_1(G_+)$ is a closed subalgebra of $L_1(G)$. Let $\hat{G}$ be the dual of G and let A' be the closed span of $\hat{G}$ in the norm of $L_\infty(G)$. Let $L_1(G)dx$ and $L_1(G_+)dx$ designate $L_1(G)$ and $L_1(G_+)$, respectively, considered as embedded (topologically) in $C_0(G)^*$.

Suppose $\mu \in M(G)$ has the property that for all $\mu_f \in L_1(G_+)dx$, $\mu * \mu_f = \mu_g$ where $\mu_g \in L_1(G_+)dx$. We will first show that μ must vanish off G_+.

Let $\{\mu_\delta\}$ be an approximate identity of $L_1(G)dx$ and let $\mu \in M(G)$. For any $h \in C_0(G)$, the set of continuous functions on G vanishing at ∞, we have

$$\mu * \mu_\delta(h) = \iint_G h(xy)\mu_\delta(dx)\mu(dy) = \iint_G h(xy)\mu_\delta(x)dx\mu(dy)$$

and

$$|\mu * \mu_\delta(h) - \mu(h)| = \left| \iint_G (h(xy) - h(x))\mu_\delta(x)dx\mu(dy) \right|.$$

Since $y \to h^y$ ($h^y(x) = h(xy)$) is uniformly continuous on G, choose V such that for $y \in V$, $\|h^y - h\|_\infty < \epsilon/\|\mu\|$. Then, taking μ_δ with support $(\mu_\delta) \subset V$ we have

$$\left| \iint_G (h(xy) - h(x))\mu_\delta(x)dx\mu(dy) \right| \leqq \|h^y - h\|_\infty \|\mu\| < \epsilon.$$

Therefore, pointwise on $C_0(G)$, $\mu * \mu_\delta \to \mu$.

Suppose that support$(\mu) \cap (G - G_+) \neq \varnothing$, and let N be a compact set with $|\mu(N)| > 0$ and $N \subset$ support$(\mu) \cap G - G_+$. (This is possible, since μ is regular and G_+ is closed.) Let $f \in C_0(G)$ be such that $f \equiv 1$ on a compact neighborhood U of N and $f \equiv 0$ off some open neighborhood W of N. Let V be a neighborhood of the identity satisfying $V \cdot N \subset U$. Choose $\{\mu_\sigma\}$ from an approximate identity of $L_1(G)dx$ with $\mu \in L_1(G_+)dx$. (G_+^0 is dense in G_+ and Haar measure is positive on open sets.) $\mu * \mu_\sigma(f) \to \mu(f)$. $\mu * \mu_\sigma$ is assumed an element of $L_1(G_+)dx$, hence is supported by G_+. Choosing $\mu_0 \in \{\mu_\sigma\}$ such that support $(\mu_0) \subset V$, then

$$|\mu * \mu_0(f)| \geqq \left| \int_{G \cap VN} 1\mu(dy) \right| \geqq |\mu(N)| > 0$$

in contradiction to the fact that support$(\mu * \mu_0) \subset G_+$. Hence any measure on G which is a multiplier of $L_1(G_+)$ must vanish off G_+.

Finally, we establish that such measures are precisely the multipliers of $L_1(G_+)$. $\Delta(L_1(G)) \subset \Delta(L_1(G_+))$, $\Delta(L_1(G))$ is the Silov boundary of $L_1(G_+)$ in $\Delta(L_1(G_+))$, and the topologies (relative and original) agree on $\Delta(L_1(G))$. See [2]. Constructing $L_1(G_+)''$ using functionals in the closed span of the Silov boundary $\Delta(L_1(G))$ in $\Delta(L_1(G_+))$, we may still reason that $L_1(G_+)^m \subset C_0(G^*)^* = L_1(G)''$. Consequently, since any multiplier of $L_1(G_+)$ must be continuous on $\Delta(L_1(G))$, we can apply Eberlein's theorem to conclude that every multiplier of $L_1(G_+)$ is in $M(G)$ and by the preceding paragraph must vanish off G_+.

THEOREM. *The multipliers of $L_1(G_+)$ are those measures in $M(G)$ with support contained in G_+.*

REFERENCES

1. R. Arens, *Operations induced on function classes*, Monatsh. Math. **55** (1951), 1–19.

2. R. Arens and I. M. Singer, *Generalized analytic functions*, Trans. Amer. Math. Soc. **81** (1956), 379–393.

3. F. T. Birtel, *Banach algebras of multipliers*, Duke Math. J. **28** (1961), 203–212.
4. ———, *Isomorphisms and isometric multipliers*, Proc. Amer. Math. Soc. **13** (1962), 204–210.
5. W. F. Eberlein, *Characterizations of Fourier-Stieltjes transforms*, Duke Math. J. **22** (1955), 465–468.
6. I. Glicksberg, *Convolution semi-groups of measures*, Pacific J. Math. **9** (1959), 51–67.
7. L. H. Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, New York, 1953.

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A GENERAL THEORY OF k-PLACE STROKE FUNCTIONS IN 2-VALUED LOGIC

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Let O be a 2-valued operator which may be applied to k ($k \geqq 1$) statements $P_1, \cdots, P_k$ to generate a new statement $O(P_1, \cdots, P_k)$. Let $o(p_1, \cdots, p_k)$ denote the truth-value function associated with $O(P_1, \cdots, P_k)$. We will assume that $o(p_1, \cdots, p_k)$ is defined by the standard truth table which is associated with the statement $O(P_1, \cdots, P_k)$ when 1 denotes "true" and 2 denotes "false." *A set of operators* $\{O_1, \cdots, O_t\}$ will be called *functionally complete* if a statement calculus based on $O_1, \cdots, O_t$ is functionally complete. If the set $\{O\}$ is functionally complete, then O will be called *a k-place stroke* and $O(P_1, \cdots, P_k)$ will be called *a k-place stroke function.* If $k=2$, there are exactly two k-place stroke functions; namely, the well-known $\uparrow(P_1, P_2)$ and $\downarrow(P_1, P_2)$ which are such that $\uparrow(p_1, p_2) = 3-\max(p_1, p_2)$ and $\downarrow(p_1, p_2) = 3-\min(p_1, p_2)$. The purpose of this paper is to give a general method for constructing and calculating the number of k-place stroke functions for any given k $(1, 2, \cdots, N)$. If $k=1$, then the set of k-place stroke functions is null. Hence, the present problem of generating k-place stroke functions becomes interesting only when $k \geqq 3$.

If a standard truth table is constructed for $O(P_1, \cdots, P_k)$, it will contain 2^k rows, and the ith row will correspond to the ith term in the lexicographical ordering of the sets of truth values $[v_1, \cdots, v_k]$ which are assigned to the arguments $P_1, \cdots, P_k$. Let X_i denote the truth value of $O(P_1, \cdots, P_k)$ in the ith row of a standard truth table for $O(P_1, \cdots, P_k)$. The truth values of $O(P_1, \cdots, P_k)$ which appear

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