FLEXIBLE ALGEBRAS OF DEGREE ONE

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1. **Introduction.** Let A be a simple, flexible, powerassociative, finite-dimensional algebra over a field of characteristic zero. Then it is known that A has a unity element 1 [**5**], and consequently A has a degree. When A has degree larger than two, Oehmke has shown [**5**] that A^+ is a simple Jordan algebra. Kokoris [**4**] has shown the same result in case A has degree two. In this paper we are able to show that if A has degree one then in fact A must be a one-dimensional algebra. Combining these results, the following theorem may be asserted.

MAIN THEOREM. *If A is a simple, flexible, powerassociative, finite-dimensional algebra of characteristic zero then A^+ is a simple Jordan algebra.*

2. PROOF. We begin with a result that is more general than actually needed to prove the main theorem.

THEOREM 1. *Let R be a flexible algebra with unity element* 1 *over a field F of characteristic not two. Suppose there exists some vector space decomposition of R, $R=F1+N$, such that for all elements a, b in N $a\cdot b=(ab+ba)/2$ is in N. Then the ideal C generated by all elements of the form $(x, y, z)=(x\cdot y)\cdot z-x\cdot(y\cdot z)$ is contained in N and hence is a proper ideal of R.*

PROOF. For arbitrary elements x_1, x_2, y in N we have $x_1y=\lambda_1 1+z_1$, and $x_2y=\lambda_2 1+z_2$, where z_1 and z_2 are in N, while λ_1, λ_2 are scalars. As in Schafer [**7**, Relation (8)] it follows from the flexible law that

$$(1)\qquad \begin{aligned}(x_1\cdot x_2)y = {} & \lambda_1x_2+\lambda_2x_1+x_1\cdot z_2+x_2\cdot z_1-(x_1\cdot y)\cdot x_2-(x_2\cdot y)\cdot x_1\\ & +(x_1\cdot x_2)\cdot y.\end{aligned}$$

As in Kokoris [**3**, p. 653] one goes on to show from (1) that

$$(2)\qquad \begin{aligned}(x_1, x_2, x_3)y = {} & (x_1, x_2, z_3)+(x_1, z_2, x_3)+(z_1, x_2, x_3)-(x_1\cdot y, x_2,x_3)\\ & -(x_1, x_2\cdot y, x_3)+(x_3\cdot y, x_2, x_1)+(x_1, x_2, x_3)\cdot y,\end{aligned}$$

where (x, y, z) is defined here as $(x, y, z)=(x\cdot y)\cdot z-x\cdot(y\cdot x)$, while $x_3y=\lambda_3 1+z_3$, where z_3 is in N and λ_3 is a scalar. Then if B is the subspace generated by all (x, y, z), relation (2) implies that $BN\subset B$

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$+B\cdot N$, $NB\subset B+B\cdot N$, and more generally $((BN)\cdots)N\subset B+B\cdot N+\cdots+((B\cdot N)\cdots)\cdot N$ etc. As a result the set C, defined as the set of all finite sums of elements from the sets B, $B\cdot N$, $(B\cdot N)\cdot N,\cdots$, can be shown to be an ideal of A. Since B is readily shown to be in N and since $N\cdot N\subset N$ by hypothesis, we may conclude that $C\subset N$. This concludes the proof of the theorem.

COROLLARY. *If R is also assumed to be simple then R^+ is an associative, commutative algebra.*

While the following theorem is not essential to the proof of the Main Theorem, it together with Theorem 1 might be useful in a study of flexible algebras where the elements of N are not necessarily nilpotent.

THEOREM 2. *If S is a flexible ring of characteristic different from two such that S^+ is powerassociative, then S must be powerassociative.*

PROOF. From the flexible law third-power associativity follows. Assume inductively k-power associativity for all $k<n$. We proceed to establish n-power associativity. The flexible law implies that

$$x^{n-1}x = (xx^{n-2})x = x(x^{n-2}x) = xx^{n-1}.$$

By a second induction suppose $x^{n-a}x^a=x^ax^{n-a}$ for $0<a<n-1$. We have already established this for $a=1$. The linearized form of the flexible law implies that

$$(x^{n-a-1}x)x^a + (x^ax)x^{n-a-1} = x^{n-a-1}(xx^a) + x^a(xx^{n-a-1}).$$

By the second induction hypothesis the first term on the left cancels the second term on the right in the last equality, leaving

$$x^{n-(a+1)}x^{(a+1)} = x^{(a+1)}x^{n-(a+1)}.$$

This completes the proof of the second induction. Powerassociativity in A^+ implies

$$(x^a\cdot x^{n-a-1})\cdot x = x^a\cdot(x^{n-a-1}\cdot x).$$

However from this it follows that

$$2x^{n-1}\cdot x = 2x^{n-a}\cdot x^a,$$

so that, for all a, $x^{n-1}\cdot x=x^{n-a}\cdot x^a$, assuming characteristic different from two. This completes the first induction and the proof of the theorem.

We note that in general powerassociativity of T^+ does not suffice to guarantee powerassociativity of T.

COROLLARY. *If R is simple then R must be powerassociative.*

Consider now the case at hand, in which A is assumed to have degree one over an algebraically closed field. Then there exists a vector space decomposition $A = F1 + N$, where in fact all elements of N are nilpotent. Albert [**2**, p. 527] has shown that in A^+, N is a subalgebra. From this one infers that A satisfies the hypotheses of Theorem **1**. From the Corollary to Theorem 1 it follows that A^+ is associative. Hence A is a noncommutative Jordan algebra. At this point a result of Schafer's [**6**, Main Theorem] may be used to conclude that A is trace-admissible. Albert [1, Principal Theorem] has shown that a trace-admissible algebra A is simple if and only if A^+ is simple. Thus A^+ is a simple, associative, commutative, finite-dimensional algebra. Then it is well known that A^+ must be a field. Therefore N must be zero. This of course means A is isomorphic to F. We have proved

THEOREM 3. *If A is a simple, flexible, powerassociative, finite-dimensional algebra over an algebraically closed field of characteristic zero and degree one then A is a one dimensional field.*

The existence of nodal, noncommutative Jordan algebras indicates that the conclusion of Theorem 3 is not true for fields of finite characteristics [**3**].

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