

ON A THEOREM OF R. JUNGEN

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Let us recall the following elementary result in the theory of analytic functions in one variable.

THEOREM (R. JUNGEN [7]). *If a is rational and b algebraic their Hadamard product c is algebraic; if, further, b is rational, c also is rational.*

For several variables, Jungen's proof shows that the theorem is still true for the Bochner-Martin [2] Hadamard product. It does not hold for the Cameron-Martin [3] and for the Haslam-Jones [6] Hadamard products. In this note we give a version of Jungen's theorem which is valid for a restricted interpretation of the notions involved when a and b are formal power series in a finite number of noncommuting variables.

1. Notations. Let R be a fixed not necessarily commutative ring with unit 1. For any finite set Z , $F(Z)$ is the free monoid generated by Z and $R_{\text{pol}}(Z)$ is the free module on $F(Z)$ over R . An element a of $R_{\text{pol}}(Z)$ will usually be written in the form $a = \sum \{ (a, f) \cdot f : f \in F(Z) \}$ where the coefficients (a, f) are in R ; $R_{\text{pol}}(Z)$ is graded in the usual manner and $\pi_n a = \sum \{ (a, f) \cdot f : f \in F(Z), \deg f \leq n \}$. We identify R with $\pi_0 R_{\text{pol}}(Z)$. $R_{\text{pol}}(Z)$ is also a ring with product $aa' = \sum \{ (a, f') (a', f'') \cdot f : f, f', f'' \in F(Z), f = f'f'' \}$.

It is well known (cf., e.g., [4; 3]) that these notions extend to the ring $R(Z)$ of the formal power series (with coefficients in R) in the noncommuting variables $z \in Z$; $R(Z)$ is topologized in the same manner as a ring of commutative formal power-series and $aa' = \lim_{n, n' \rightarrow \infty} (\pi_n a)(\pi_{n'} a')$. Any $b \in R^*(Z) = \{ a \in R(Z) : \pi_0 a = 0 \}$ has a quasi-inverse $(-b)^* = \lim_{n \rightarrow \infty} \sum_{n' < n} (-b)^{n'}$. If a is invertible, $a^{-1} = (1 + b^*)(\pi_0 a^{-1})$ where $b = -(\pi_0 a^{-1})(a - \pi_0 a) \in R^*(Z)$. We shall say that $S^* \subset R^*(Z)$ is *rationally closed* if $r, r' \in R$, $b, b' \in S^*$ imply $rb + b'r', bb', b^* \in S^*$. If this is so, the set of those elements a of $R(Z)$ such that $a - \pi_0 a \in S^*$ is a ring containing the inverses of its invertible elements.

DEFINITION 1. $R_{\text{rat}}^*(X)$ is the least rationally closed subset (of $R(X)$) containing X .

Now let $Y = \{y_j\}$ be a set of a finite number M of new variables and $R^M(X \cup Y)$ (resp. $R_{\text{pol}}^M(X \cup Y)$) the cartesian product of M copies

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of the R -module $R(X \cup Y)$ (resp. $R_{\text{pol}}^M(X \cup Y)$). For each $q = (q_1, \dots, q_m) \in R^M(X \cup Y)$, $\pi_n q = (\pi_n q_1, \dots, \pi_n q_m)$. If $q \in R^{*M}(X \cup Y)$ (i.e., if $\pi_0 q = 0$) let λ_q be the homomorphism of the monoid $F(X \cup Y)$ into the multiplicative monoid structure of $R(X \cup Y)$ that is induced by $\lambda_q x = x$ if $x \in X$ and $\lambda_q y_j = q_j$ if $y_j \in Y$. Since $\pi_0 q = 0$, λ_q can be extended to an endomorphism of the R -module $R(X \cup Y)$ by $\lambda_q a = \sum \{(a, f) \lambda_q f : f \in F(X \cup Y)\}$; also, $\lambda_q p = (\lambda_q p_1, \dots, \lambda_q p_M)$ for any $p \in R^M(X \cup Y)$.

We shall say that $p \in R^{*M}(X \cup Y)$ is a *proper system* if $(p_j, \gamma_{j'}) = 0$ for all $j, j' \leq M$. Then, if $q \in R^{*M}(X)$, $\lambda_q p \in R^{*M}(X)$ and $\pi_{n+1} \lambda_q p = \pi_{n+1} \lambda_{\pi_n q} p$ for all n . Consider now the infinite sequence $p(0) = 0$, $p(1) = \lambda_{p(0)} p$, \dots , $p(m+1) = \lambda_{p(m)} p$, \dots . Trivially, $\pi_{m'} p(m') = \pi_{m'} p(m' + m'') \in R^{*M}(X)$ for $m' = 0$ and all m'' . If these relations hold for $m' \leq m$, they still hold for $m+1$ because

$$\begin{aligned} \pi_{m+1} p(m+1) &= \pi_{m+1} \lambda_{p(m)} p = \pi_{m+1} \lambda_{\pi_m p(m)} p = \pi_{m+1} \lambda_{\pi_m p(m+m'')} p \\ &= \pi_{m+1} \lambda_{p(m+m'')} p = \pi_{m+1} p(m+1+m''). \end{aligned}$$

Hence, $p(\infty) = \lim_{m \rightarrow \infty} p(m)$ exists and it satisfies $p(\infty) \in R^{*M}(X)$, $\pi_0 p(\infty) = 0$, $p(\infty) = \lambda_{p(\infty)} p$. In fact, $p(\infty)$ is the only element to satisfy these equations because if $\pi_0 p' = 0$ and $p' = \lambda_{p'} p$, any relation $\pi_m p(\infty) = \pi_m p'$ implies $\pi_{m+1} p' = \pi_{m+1} \lambda_{\pi_m p'} p = \pi_{m+1} \lambda_{\pi_m p(\infty)} p = \pi_{m+1} p(\infty)$. For this reason we call $p(\infty)$ the *solution* of p .

DEFINITION 2. $R_{\text{alg}}^*(X)$ is the least subset (of $R^*(X)$) that contains every coordinate of the solution of any proper system having its coordinates in $R_{\text{pol}}^*(X \cup Y)$.

(REMARK. It can easily be shown that $R_{\text{alg}}^*(X)$ is rationally closed and that it contains every coordinate of the solution of any proper system having its coordinates in $R_{\text{alg}}^*(X \cup Y)$.)

DEFINITION 3. For any

$$a, b \in R(X), \quad a \odot b = \sum \{(a, f)(b, f) \cdot f : f \in F(X)\}.$$

2. Main result.

Property 2.1. The element a of $R^*(X)$ belongs to $R_{\text{rat}}^*(X)$ if and only if there exists a finite integer $N \geq 2$ and a homomorphism μ of $F(X)$ into the multiplicative monoid of $R^{N \times N}$ (the ring of the $N \times N$ matrices with entries in R) such that $a = \sum \{\mu f_{1,N} \cdot f : f \in F(X)\}$ (abbreviated as $\sum \mu f_{1,N} \cdot f$).

PROOF. (1) *The condition is necessary.* This is trivial if $a = \pi_1 a$. Hence it suffices to show that for any $r, r' \in R$, $a = \sum \mu f_{1,N} \cdot f$ and $a' = \sum \mu' f_{1,N'} \cdot f$ one can construct suitable homomorphisms giving $ra + a'r'$, aa' and a^* . This is done below, defining the homomorphisms by their restriction to X .

Addition. Let $N'' = N + N' + 2$ and $\mu''x \in R^{N'' \times N''}$ defined for each $x \in X$ by

$$\begin{aligned}\mu''x_{i,1} &= \mu''x_{N'',i} = 0 & \text{for } 1 \leq i \leq N''; \\ \mu''x_{1,i+1} &= r\mu x_{1,i} \text{ and } \mu''x_{i+1,N''} = \mu x_{i,N} & \text{for } 1 \leq i \leq N; \\ \mu''x_{1,i+N+1} &= \mu'x_{1,i} \text{ and } \mu''x_{i+N+1,N''} = \mu'x_{i,N'} \cdot r' & \text{for } 1 \leq i \leq N'; \\ \mu''x_{i,i'} &= \text{the direct sum of } \mu x \text{ and } \mu'x & \text{for } 2 \leq i, i' \leq N'' - 1; \\ \mu''x_{1,N''} &= r\mu x_{1,N} + \mu'x_{1,N'} \cdot r'.\end{aligned}$$

The verification is trivial.

Product. Let $N'' = N + N'$ and define $\nu f \in R^{N'' \times N''}$ for each $f \in F(X)$ by $\nu f_{i,i'} = \mu f_{i,N}$ if $f \neq 1$, $1 \leq i \leq N$, $i' = N + 1$; $\nu f_{i,i'} = 0$, otherwise. Then, if $\mu''x = \bar{\mu}x + \nu x$ where $\bar{\mu}x$ is the direct sum of μx and $\mu'x$, one has for each $f = x^{(1)}x^{(2)} \cdots x^{(n)}$, $\mu''f = \bar{\mu}f + \sum \{\bar{\mu}f' \nu x^{(i)} \bar{\mu}f'' : f'x^{(i)}f'' = f\}$. Since $\nu f x^{(i)} = \bar{\mu}f \nu x^{(i)}$ and $(\nu f'' \bar{\mu}f')_{1,N''} = 0$ when $f'' = 1$, one has $\mu''f_{1,N''} = \sum \{(\mu f'_{1,N})(\mu'f''_{1,N'}) : f'f'' = f\}$. Hence, $\sum \mu''f_{1,N''} \cdot f = aa'$.

Quasi-inverse. Let $N'' = N$ and define $\nu f \in R^{N \times N}$ for each $f \in F(X)$ by $\nu f_{i,i'} = \mu f_{i,N}$ if $f \neq 1$, $1 \leq i \leq N$, $i' = 1$; $\nu f_{i,i'} = 0$, otherwise. Then $\mu''x = \mu x + \nu x$ and since $\mu f \nu x = \nu f x$ identically one has $\mu''f = \sum \nu f^{(1)} \nu f^{(2)} \cdots \nu f^{(k)} \mu f^{(k+1)}$ where the summation is over all the factorisations $f = f^{(1)}f^{(2)} \cdots f^{(k+1)}$ of f in an arbitrary number of factors. The $(1, N)$ entry of any of these products is zero unless all its factors are different from 1 and under this condition, it is equal to $\mu f^{(1)}_{1,N} \mu f^{(2)}_{1,N} \cdots \mu f^{(k+1)}_{1,N}$. Hence, $\sum \mu''f_{1,N} \cdot f = \sum_{n>0} a^n = a^*$ and the first part of the proof is completed.

(2) *The condition is sufficient.* We say that the proper system p is linear if for each $j \leq M$, $p_j = q_{j,0} + \sum_{j'} q_{j,j'} y_{j'}$ where all the q 's belong to $R_{\text{rat}}^*(X)$ and we verify that all coordinates of the solution of such a system belong to $R_{\text{rat}}^*(X)$.

This is trivial if $M = 1$ because $p(\infty) = (1 - q_{1,1})^{-1} q_{1,0} = (1 + q_{1,1}^*) q_{1,0}$. If it is true for $M' < M$ it is still true for M . Indeed, because $p(\infty)_M = (1 - q_{M,M})^{-1} (q_{M,0} + \sum_{j < M} q_{M,j'} p(\infty)_{j'})$, the proper linear system p' defined by $p'_j = p_j - q_{j,M} y_M + q_{j,M} p_M$ for $j < M$ and $p'_M = (1 - q_{M,M})^{-1} (p_M - q_{M,M} y_M)$ is such that $p(\infty) = p'(\infty)$. Since its first $M - 1$ coordinates do not involve y_M the result follows from the induction hypothesis.

Now, given a homomorphism μ of $F(X)$ into $R^{M \times M}$, the M elements $a_j = \sum \{\mu f_{j,M} \cdot f : f \in F(X), f \neq 1\}$ are such that $(a_j, xf) = \sum_{j'} \mu x_{j,j'}(a_{j'}, f)$. Hence (a_1, \cdots, a_M) is the solution of the linear proper system such that $q_{j,0} = \sum \{\mu x_{j,M} \cdot x : x \in X\}$, $q_{j,j'} = \sum \{\mu x_{j,j'} \cdot x : x \in X\}$ for each j, j' and 2.1 is proved.

We now consider two subrings R' and R'' of R that commute element-wise.

Property 2.2. If $a = \sum \mu' f_{1,N} \cdot f \in R'_{\text{rat}}^*(X)$ where μ' is a homomorphism into $R'^{N \times N}$ and if $b = p(\infty)_1 \in R''_{\text{alg}}^*(X)$ where the proper system p has its coordinates in $R''_{\text{pol}}^*(X \cup Y)$, then $a \odot b \in R_{\text{alg}}^*(X)$. If, further, $b \in R'_{\text{rat}}^*(X)$ then $a \odot b \in R_{\text{rat}}^*(X)$.

PROOF. We verify first the case of $b \in R'_{\text{rat}}^*(X)$, i.e., of $b = \sum \mu'' f_{1,N''} \cdot f$ for some N'' and μ'' . Then $a \odot b = \sum (\mu' \otimes \mu'') f_{1,NN''} \cdot f$ where the kroneckerian product $\mu' \otimes \mu''$ is a homomorphism of $F(X)$ into $R^{NN'' \times NN''}$ because R' and R'' commute and the result is proved.

For the general case we denote by $K(Z)$ for any set Z the ring of the $N \times N$ matrices with entries in $R(Z)$. We shall have to consider several homomorphisms of module $\sigma: R^M(Z') \rightarrow K^M(Z'')$ where Z' and Z'' are two finite sets. In each case σ is defined by a mapping $Z' \rightarrow K(Z'')$ which is extended in a natural fashion to a homomorphism of the monoid $F(Z')$ into the multiplicative structure of $K(Z'')$. Then for each

$$a = (a_1, \dots, a_M) \in R^M(Z'), \quad \sigma a_j = \sum \{(a_j, g) \cdot \sigma g : g \in F(Z')\}$$

and $\sigma a = (\sigma a_1, \dots, \sigma a_M)$.

More specifically, $\mu: R^M(X) \rightarrow K^M(X)$ is induced by a mapping $\mu: X \rightarrow K(X)$ such that the entries of each μx belong to $R'^*(X)$.

For each $q \in R''^{*M}(X)$, $\lambda_{\mu q}: R(X \cup Y) \rightarrow K^M(X)$ is induced by $\lambda_{\mu q} f = \mu f$ if $f \in F(X)$ and $\lambda_{\mu q} y_j = \mu q_j$ if $y_j \in Y$. Hence, since R' and R'' commute element-wise, $\mu \lambda_{\mu q} g = \lambda_{\mu q} g$ for each $g \in F(X \cup Y)$ (with $\lambda_{\mu q}$ as previously defined). Consequently, $\mu \lambda_{\mu q} p = \lambda_{\mu q} p$ for any $p \in R''^{*M}(X \cup Y)$.

Let now $Z = \{z_{j,i,i'}\} (1 \leq j \leq M; 1 \leq i, i' \leq N)$, a set of $M \times N \times N$ new variables and $\nu: R^M(X \cup Y) \rightarrow K^M(X \cup Z)$ induced by $\nu f = \mu f$ if $f \in F(X)$, $\nu y_j =$ the $N \times N$ matrix with entries $z_{j,i,i'}$ if $y_j \in Y$. Also $\lambda_{\nu q}: R(X \cup Z) \rightarrow R(X)$ is induced by $\lambda_{\nu q} f = f$ if $f \in F(X)$ and $\lambda_{\nu q} z_{j,i,i'} = (\nu q_j)_{i,i'}$ if $z_{j,i,i'} \in Z$. We extend $\lambda_{\nu q}$ to a homomorphism $K^M(X \cup Z) \rightarrow K^M(X)$ by defining $\lambda_{\nu q} m$ for any $m \in K(X \cup Z)$ as the $N \times N$ matrix with entries $\lambda_{\nu q}(m_{i,i'})$.

Because R' and R'' commute, $\lambda_{\mu q} g = \lambda_{\nu q} \nu g$ for each $g \in F(X \cup Y)$ and, consequently, $\lambda_{\mu q} p = \lambda_{\nu q} \nu p$ for each $p \in R''^{*M}(X \cup Y)$. Hence, if p is a proper M -dimensional system with coordinates in $R''^*(X \cup Y)$ we have $\mu p(\infty) = \mu \lambda_{\mu p(\infty)} p = \lambda_{\mu p(\infty)} p$. Since μ and ν coincide on $R''^{*M}(X)$, we have also $\mu p(\infty) = \nu p(\infty) = \lambda_{\nu p(\infty)} p = \lambda_{\nu p(\infty)} \nu p$.

However, the $M \times N \times N$ elements $p'_{j,i,i'} = (\nu p_j)_{i,i'}$ all belong to $R^*(X \cup Z)$ and they constitute a proper system p' of dimension MN^2 . Thus, by construction, $(\mu p(\infty))_{i,i'} = p'(\infty)_{j,i,i'}$ identically. If, fur-

ther, $p \in R_{\text{pol}}'^{*M}(X \cup Y)$ all the entries appearing in νp belong to $R_{\text{pol}}^*(X \cup Z)$ and then finally $(\mu p(\infty))_{i,i'} \in R_{\text{alg}}^*(X)$.

This completes the proof because

$$\begin{aligned} a \odot b &= \sum \{ (b, f) \mu' f_{1,N} : f \in F(X) \} \\ &= \sum \{ (b, f) \mu f_{1,N} : f \in F(X) \} = \mu b_{1,N} \end{aligned}$$

where for each $x \in X$, μ is defined by $\mu x_{i,i'} = \mu' x_{i,i'} \cdot x$.

REMARK 1. Definitions 1, 2, and 3 and the computations of this section used only the structure of monoid of the additive groups considered. Hence, the results are still valid when an arbitrary *semi-ring* S is taken in place of R . For S consisting of two Boolean elements, Jungen's theorem and its special case for b rational have been obtained in a different form by Y. Bar-Hillel, M. Perles and E. Shamir [1] (also by S. Ginsburg and G. F. Rose [5]) and by S. Kleene [8] respectively as by-products of more sophisticated theories.

REMARK 2. Let $R = C$, the field of complex numbers; and p a proper system of dimension M . Introducing $4M$ new symbols z_j and replacing each y_j by $z_{4j} + iz_{4j+1} - z_{4j+2} - iz_{4j+3}$ in the p_j s we can deduce from p a new system of dimension $4M$ in which all the coefficients are non-negative real numbers and whose solution is simply related to $p(\infty)$.

Assume now that $p \in C_{\text{pol}}^{*M}(X \cup Y)$ has only real non-negative coefficients and denote by α a homomorphism of $C_{\text{pol}}(X \cup Y)$ into C . Because of the assumption that $(p_j, y_j) = (p_j, 1) = 0$, identically, we can find an $\epsilon > 0$ such that $|\alpha p_j| < \epsilon$ for all j when $|\alpha x| \leq \epsilon$ and $|\alpha y| \leq 2\epsilon$ for all $x \in X$ and $y \in Y$. Since the sequence $\alpha p(0), \alpha p(1), \dots, \alpha p(n), \dots$ is monotonically increasing it converges to a finite solution (cf., e.g., [10]).

Hence, the canonical epimorphism of $C_{\text{pol}}(X \cup Y)$ onto the ring of the ordinary (commutative) polynomials can be extended to an epimorphism of $C_{\text{alg}}(X)$ onto the ring of the Taylor series of the algebraic functions.

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