ABSOLUTE CONTINUITY OF CERTAIN UNITARY AND HALF-SCATTERING OPERATORS

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1. **The theorem.** *Let A be a self-adjoint operator, bounded from below, and D a bounded, non-negative self-adjoint operator on a Hilbert space $\mathfrak{H}$ for which the set*

$$\mathfrak{D}_A \cap \mathfrak{R}_{D^{1/2}} \text{ is dense.} \tag{1}$$

Suppose that $A+D$ is unitarily equivalent to A and that U is any unitary operator effecting this equivalence, thus

$$A + D = UAU^*. \tag{2}$$

Then U is absolutely continuous, that is, if U has the spectral resolution

$$U = \int_0^{2\pi} e^{i\lambda} dE(\lambda), \tag{3}$$

and if x is an arbitrary element of the Hilbert space, then $\|E(\lambda)x\|^2$ is an absolutely continuous function of λ.

If A is bounded, then $\mathfrak{D}_A = \mathfrak{H}$ and (1) reduces to the assumption that 0 cannot be in the point spectrum of D. In this case, the assertion of the theorem was proved in [2]. In [3], there were obtained lower estimates for the measure of the spectrum of U both when A was bounded and also in the case when A was supposed only half-bounded. It will be shown in the present paper that the methods used in this latter case will also yield the absolute continuity of U under the conditions specified in the theorem.

An application to half-scattering operators will be given in §3.

2. **Proof of the theorem.** Since (1) and (2) hold if A is replaced by $A+cI$, where $c=\text{const.}$, it is clear that there is no loss of generality in assuming that

$$A \geqq 0. \tag{4}$$

If $f(\lambda)$ is a real-valued function of period 2π with a continuous first derivative and having the Fourier series

$$f(\lambda) = \sum_{k=-\infty}^{\infty} c_k e^{ik\lambda}, \qquad c_k = (2\pi)^{-1}\int_0^{2\pi} f(\lambda)e^{-ik\lambda}d\lambda, \qquad c_{-k} = \bar{c}_k, \tag{5}$$

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then it follows from (3) and (5) that

$$D^{1/2}\int_0^{2\pi} f(\lambda)dE(\lambda) = c_0D^{1/2} + \sum_{k=1}^{\infty} c_kD^{1/2}U^k + \sum_{k=1}^{\infty} \bar{c}_kD^{1/2}U^{*k}.\text{[2]} \tag{6}$$

Next, let y be in $\mathfrak{D}_A \cap \mathfrak{R}_{D^{1/2}}$, so that $y = D^{1/2}x$ is in $\mathfrak{D}_A$. It follows from (6) that

$$\begin{aligned}&\left(x, D^{1/2}\int_0^{2\pi} f(\lambda)dE(\lambda)y\right)\\ &\qquad = (x, c_0D^{1/2}y) + 2\,\mathrm{Re}\left(x, \sum_{k=1}^{\infty} c_kD^{1/2}U^ky\right).\end{aligned} \tag{7}$$

Since $(\mathrm{Re}(\cdots))^2 \leqq \|x\|^2(\sum_{k=1}^{\infty} |c_k|^2)(\sum_{k=1}^{\infty} \|D^{1/2}U^ky\|^2)$, an application of the Schwarz inequality to (7) yields

$$\begin{aligned}&\left(\int_0^{2\pi} f(\lambda)d\|E(\lambda)y\|^2\right)^2\\ &\qquad \leqq 2\left[|c_0|^2\|y\|^4 + 4\|x\|^2\left(\sum_{k=1}^{\infty}|c_k|^2\right)\left(\sum_{k=1}^{\infty}\|D^{1/2}U^ky\|^2\right)\right].\end{aligned} \tag{8}$$

But (2) implies that for $n = 1, 2, \cdots, \sum_{k=1}^{n} U^{*k}DU^k = A - U^{*n}AU^n \leqq A$, the inequality by (4), and so $\sum_{k=1}^{\infty}\|D^{1/2}U^ky\|^2 \leqq (Ay, y)$.

Relation (8) and the Parseval relation $(2\pi)^{-1}\int_0^{2\pi} f^2(\lambda)d\lambda = |c_0|^2 + 2\sum_{k=1}^{\infty}|c_k|^2$ now imply

$$\left(\int_0^{2\pi} f(\lambda)d\|E(\lambda)y\|^2\right)^2 \leqq C(x)\int_0^{2\pi} f^2(\lambda)d\lambda, \tag{9}$$

where $C(x)$ is a number which depends on x (and $y = D^{1/2}x$) but not on the choice of $f(\lambda)$. Since (9) holds for every smooth function on $[0, 2\pi]$ satisfying $f(0) = f(2\pi)$ and since ordinary Lebesgue measure is absolutely continuous, it follows by a standard argument (involving approximations of characteristic functions of intervals by smooth functions $f(\lambda)$ satisfying $f(0) = f(2\pi)$) that $\|E(\lambda)y\|^2$ is also absolutely continuous. It then follows from (1) that $\|E(\lambda)x\|^2$ is absolutely continuous for *all* x in the Hilbert space and the proof of the theorem is complete.

3. **Half-scattering operators.** Let A denote the quantum mechanical (half-bounded) energy operator $-d^2/dx^2$ on the Hilbert space

[2] Since $f(\lambda)$ has a continuous derivative, then $\|\sum_{k=-\infty}^{\infty} c_kD^{1/2}U^k\| \leqq (\sum_{k=-\infty}^{\infty}|c_k|)\|D^{1/2}\| < \infty$, and so the summations of (6) converge in the uniform norm topology. The author is indebted to the referee for this observation.

$L^2(-\infty, \infty)$ and D be a perturbation potential $V(x)$, where $V(x)$ is continuous and satisfies

$$(10) \qquad 0 \leqq V(x) < \text{const.}, \qquad -\infty < x < \infty,$$

and

$$(11) \qquad \int_{-\infty}^{\infty} V(x)dx < \infty.$$

Then A and $A+D$ are absolutely continuous and each has the half-line $0 \leqq \lambda < \infty$ as spectrum (cf. [4] and the references there to Weyl, Kodaira, Titchmarsh). Moreover, it follows from results of Kuroda [1] (cf. [4]) that the half-scattering operators

$$(12) \qquad U_+ = \lim_{t\to\infty} U_t \text{ and } U_- = \lim_{t\to-\infty} U_t, \text{ where } U_t = e^{it(A+D)}e^{-itA},$$

exist as strong limits satisfying (2). As a consequence of the theorem of the present paper, it follows that if, in addition to (10) and (11), $V(x)$ satisfies

$$(13) \qquad V(x) > 0 \text{ almost everywhere on } (-\infty, \infty),$$

then the half-scattering operators U_+ and U_- of (12) are absolutely continuous. In fact, (10) implies that $D = V(x)$ is bounded and non-negative while (13) implies (1).

It can be noted that if, in addition to (10), (11) and (13), also

$$(14) \qquad \liminf\, (b-a)^{-3}\int_a^b V^{-1}(x)dx = 0, \quad \text{as} \quad b - a \to \infty,$$

is assumed, then, as was shown in [4], the (absolutely continuous) spectrum of each of the operators U_+ and U_- must be the entire unit circle $|z| = 1$.

References

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