ON RINGS OF WHICH ANY ONE-SIDED QUOTIENT RINGS ARE TWO-SIDED

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0. **Introduction and preliminaries.** Following R. E. Johnson [1] we assume in this paper that any rings we shall be concerned with satisfy either one or both of the following conditions:

(J_l) If the right annihilator of a left ideal A is nonzero, then there exists a nonzero left ideal B such that $A \cap B = 0$.

(J_r) = the right left symmetry of (J_l).[1]

We say that a ring is a J_l-ring, a J_r-ring or a J-ring if it satisfies (J_l), (J_r) or both of them.

A module A is called an essential extension of a submodule B if $B \cap C \neq 0$ for every nonzero submodule C of A. A module is said to be injective if it is a direct summand of every extension module. It is well known that every module M has a maximal essential extension $\hat{M}$. $\hat{M}$ is injective, and is unique to within an isomorphism over M.

Let S be a J_l-ring. Then we can define the multiplication in the maximal essential extension $\hat{S}$ of the left S-module S such a way that (i) $\hat{S}$ forms a ring and (ii) the multiplication coincides, on $S \times \hat{S}$, with the scalar multiplication. This ring is unique up to an isomorphism over S, and is denoted by $\bar{S}_l$. As is known, $\bar{S}_l$ is regular (in the sense of von Neumann), and is left self injective, that is, injective as a left module over itself. An extension ring T of a J_l-ring S is called a left quotient ring of S if the left S-module T is an essential extension of the left S-module S. It is also known that every left quotient ring of S is isomorphic, over S, to a subring of $\bar{S}_l$. Thus, $\bar{S}_l$ is the maximal left quotient ring of S.

We define similarly a right quotient ring and the maximal right quotient ring $\bar{S}_r$ of a J_r-ring S.

For any J-ring S it is easily seen that the following conditions are equivalent:

(i) There exists an extension ring T of S with the properties that (a) it is regular (both left and right) self injective, and (b) every nonzero one-sided S-submodule of T has a nonzero intersection with S.

(ii) Every left quotient ring of S is a right quotient ring of S, and every right quotient ring of S is a left quotient ring of S.

In this case any maximal left quotient ring of S and any maximal

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[1] In the terminology of Johnson [1] (J_l) ((J_r)) means that the left (right) singular ideal of the ring is zero.

right quotient ring of S are mutually isomorphic over S. We denote this fact by writing $\overline{S}_l=\overline{S}_r$.

The main theorem (Theorem 3.3) in this paper states that for any J-ring S we have $\overline{S}_l=\overline{S}_r$ if and only if S satisfies the converse of (J_l) and (J_r), that is, the following two conditions:

(K_l) If $A\cap B=0$, $B\neq 0$, for left ideals A and B, then the right annihilator of A is nonzero;

(K_r) = the right left symmetry of (K_l).

1. **Strongly regular rings.** A ring is called strongly regular if for any element x there exists an element y such that $x^2y=x$. As is well known, in this case xy is central idempotent and $xy=yx$. Every one-sided ideal of a strongly regular ring is two-sided. A regular ring is strongly regular if and only if it is of index 1, that is, it contains no nonzero nilpotent elements.

A left ideal A of a ring is said to be closed if there are no left ideals B such that $A\neq B$ and B is an essential extension of A. The set of all closed left ideals of a J_l-ring S forms a complete complemented modular lattice, which is denoted by $L(S)$. If a ring T is a left quotient ring of a J_l-ring S, then T is also a J_l-ring, and $L(T)$ is isomorphic to $L(S)$ by the correspondence $A(\in L(T))\to A\cap S$. A left ideal of $\overline{S}_l$ is closed if and only if it is principal. Thus, $L(\overline{S}_l)$ is the lattice of all principal left ideals of $\overline{S}_l$. Let A, $B\in L(S)$. Then $A\cap B$ is the intersection of A and B, while $A\cup B$ is the maximal essential extension of $A+B$.

We can define similarly the notion of a closed right ideal. $R(S)$ denotes the lattice of all closed right ideals of a J_r-ring S. $R(\overline{S}_r)$ is the lattice of principal right ideals of $\overline{S}_r$.

THEOREM 1.1. *Let S be a regular ring. Then the following conditions are equivalent*:

(i) *S is strongly regular.*

(ii) *$L(S)$ is distributive.*

(iii) *The lattice L of principal left ideals of S is distributive.*

PROOF. (i)⇒(ii). Let A, $B\in L(S)$. Then $AB\subset A\cap B=(A\cap B)^2\subset AB$, and so $AB=A\cap B$. Thus, for any P, Q, $R\in L(S)$ we have $P\cap(Q+R)=P(Q+R)=PQ+PR=(P\cap Q)+(P\cap R)$. Since $L(S)$ is complemented, $P\cap(Q\cup R)=((P\cap Q)\cup(P\cap R))\cup C$ and $((P\cap Q)\cup(P\cap R))\cap C=0$ for some $C\in L(S)$. Then $((P\cap Q)+(P\cap R))\cap C=0$ and $P\cap(Q+R)\cap C=0$, hence $P\cap(Q\cup R)\cap C=0$, which means that $C=0$ and $P\cap(Q\cup R)=(P\cap Q)\cup(P\cap R)$.

(ii)⇒(iii) follows immediately since L is a sublattice of $L(S)$.

(iii)⇒(i). Let $Se=Sf$, $e=e^2$ and $f=f^2$. Then $S(1-e)=S(1-f)$ by (iii). Thus $e=f$. This shows that every idempotent is central. Therefore $xyx=x$ means $x^2y=x$, as desired. In case $S\not\ni 1$, it follows from the above argument tSt is strongly regular for any $t=t^2$. Let $S\ni x$, and let a, b, c be idempotents such that $Sx=Sa$, $xS=bS$ and $aS+bS=cS$. Then $x\in tSt$, $t=t^2$ for $t=a+c-ac$. Therefore S is strongly regular.

COROLLARY 1.2. *If S is strongly regular, then so is $\bar{S}_l$.*

In fact, $L(S)$ is distributive by assumption. Since $L(S)\simeq L(\bar{S}_l)$, $L(\bar{S}_l)$ also is distributive, and hence $\bar{S}_l$ is strongly regular.

COROLLARY 1.3. *Let S be a J_l-ring. Then $\bar{S}_l$ is strongly regular if and only if $L(S)$ is distributive.*

In fact, $L(\bar{S}_l)$ is distributive if and only if so is $L(S)$.

THEOREM 1.4. *Let S be a J-ring. If S has a strongly regular left quotient ring P and a strongly regular right quotient ring Q, then $\bar{S}_l=\bar{S}_r$.*

PROOF. Let $0\neq p\in P$. Then $ap=b\neq 0$ for some $a, b\in S$, and $a^2q=a$ for some $q\in Q$. Since $b\neq 0$, $0\neq b^2=bap$ and $ba\neq 0$. Hence $0\neq(ba)^2=b(ab)a$ and $0\neq ab=a^2qb=a(aq)b=a(aq)^2b=a^2(qaq)b=a^2(aq^2)b$, whence $aq^2b\neq 0$. Thus, $(aq^2b)c=d\neq 0$ for some $c, d\in S$. $abc=(a^2q)bc=a(aq)bc=a(aq)^2bc=a^2(qaqb)c=a^2(aq^2b)c=a^2d$, and so $a^2pc=a(ap)c=a^2d$. Hence $a^2(pc-d)=0$ and $((pc-d)a^2)^2=0$, which implies that $(pc-d)a^2=0$ and $p(ca^2)=da^2$. Now $0\neq d=aq^2bc=(aq)(qbc)=(aq)^3(qbc)=(aq)(qbc)(aq)^2=(aq^2bc)a^2q^2=da^2q^2$, and hence $da^2\neq 0$. This shows that P is a right quotient ring of S. Thus we have proved that every strongly regular left quotient ring of S is a right quotient ring of S provided S has at least one strongly regular right quotient ring. It is not too hard to see that $\bar{S}_l$ is the maximal left quotient ring of P. Hence $\bar{S}_l$ is strongly regular by Corollary 1.2. Therefore $\bar{S}_l$ is a right quotient ring of S. This implies that every left quotient ring of S is a right quotient ring of S. Similarly we can show that every right quotient ring of S is a left quotient ring of S. Thus, $\bar{S}_l=\bar{S}_r$, completing the proof.

2. **The condition** (K). For any subset Z of an extension ring of a ring S we denote by $l(S, Z)$ the set of all $x\in S$ such that $xZ=0$. Similarly $r(S, Z)$ denotes the right annihilator, in S, of Z.

LEMMA 2.1. *Let S be a J_l-ring, and let $A\in L(\bar{S}_l)$. Then $r(S, A\cap S)=r(S, A)$.*

PROOF. $\supset$ is obvious. Let $(A\cap S)x=0$, $x\in S$, and set $B=l(\bar{S}_l, x)$. Since $B\in L(\bar{S}_l)$ and $B\cap S\supset A\cap S$, we have $B\supset A$ by the isomorphism $L(S)\simeq L(\bar{S}_l)$. Hence $Ax=0$ and $x\in r(S, A)$. Therefore $r(S, A\cap S)\subset r(S, A)$.

THEOREM 2.2. *Let S be a J_l-ring. Then the following conditions are equivalent*:

(K_l) *If $A \cap B = 0$, $B \neq 0$ for two left ideals A and B, then the right annihilator of A is nonzero.*

(K_l') *Every closed left ideal is an annihilator.*

(K_l'') *Every nonzero right ideal of $\bar{S}_l$ has a nonzero intersection with S.*

PROOF. $(K_l) \Rightarrow (K_l'')$. Suppose that $S \cap D = 0$ for some nonzero right ideal D of $\bar{S}_l$. Let $0 \neq e = e^2 \in D$. By Lemma 2.1 $r(S, \bar{S}_l(1-e) \cap S) = r(S, \bar{S}_l(1-e)) = r(\bar{S}_l, \bar{S}_l(1-e)) \cap S = e\bar{S}_l \cap S \subset D \cap S = 0$. Since $(\bar{S}_l(1-e) \cap S) \cap (\bar{S}_l e \cap S) = 0$ and $\bar{S}_l e \cap S \neq 0$, this contradicts (K_l).

$(K_l'') \Rightarrow (K_l')$. Let A be a closed left ideal, and set $l(S, r(S, A)) = B$. If $A \neq B$, there is a left ideal $C \neq 0$ such that $B \supset C$ and $A \cap C = 0$. Let D be a maximal left ideal such that $D \supset A$ and $D \cap C = 0$. Then D is closed, and $D = \bar{S}_l e \cap S$ for some $e = e^2 \neq 1$. By $(K_l'')(1-e)\bar{S}_l \cap S \neq 0$. Let $0 \neq x \in (1-e)\bar{S}_l \cap S$. Then $Dx = 0$ and $Ax = 0$, hence $Bx = 0$ and $Cx = 0$. Thus $(D \oplus C)x = 0$, and $(D \oplus C) \cap E = 0$ for some left ideal $E \neq 0$. Hence $C \cap (D \oplus E) = 0$, which contradicts the maximality of D. Therefore $A = B$.

$(K_l') \Rightarrow (K_l)$. Let $A \cap B = 0$ and $B \neq 0$ for left ideals A, B of S. Let C be a maximal left ideal such that $C \supset A$ and $C \cap B = 0$. Then C is closed, and hence is an annihilator by (K_l'). If $r(S, A) = 0$, then $r(S, C) = 0$ and $C = S$, which means that $B = 0$, a contradiction. Thus $r(S, A) \neq 0$, as desired.

COROLLARY 2.3. *$\bar{S}_l$ satisfies (K_l) for any J_l-ring S.*

In fact, $\bar{S}_l$ satisfies (K_l'') because $\bar{S}_l$ is the maximal left quotient ring of itself.[2]

COROLLARY 2.4. *If a J-ring S satisfies (K_l) and (K_r), then $L(S)$ is dually isomorphic to $R(S)$ by the annihilator relation.*

This is evident from (K_l') and its symmetry.[3]

COROLLARY 2.5. *Let S be a J_l-ring, and T a left quotient ring of S.*

(i) *If S satisfies (K_l), then so does T.*

(ii) *If T is a right quotient ring of S, and if T satisfies (K_l), then S also satisfies it.*

In fact, $\bar{S}_l$ is the maximal left quotient ring of T. Let A be a nonzero right ideal of $\bar{S}_l$. If S satisfies (K_l), $A \cap S \neq 0$ by (K_l''). Hence

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[2] Of course, (K_l) itself also follows immediately from the injectivity of $\bar{S}_l$.

[3] It should be noted that every annihilator left (right) ideal of a $J_l-(J_r-)$ ring is closed.

$A \cap T \neq 0$, and T also satisfies (K_l'). Conversely, if T is a right quotient ring of S, $A \cap T \neq 0$ implies that $A \cap S \neq 0$. This proves (ii).

LEMMA 2.6. *Let S be a J_l-ring, and let $S = A \oplus B$, A and B being ideals. If S satisfies (K_l), then so does A.*

PROOF. Let P, Q be left ideals of A such that $P \cap Q = 0$, $Q \neq 0$. Then $(P \oplus B) \cap Q = 0$. Hence $(P \oplus B)(a+b) = 0$, $a+b \neq 0$ for some $a \in A$, $b \in B$. Therefore $Sb = 0$ and $b = 0$, whence $a \neq 0$. Since $Pa = 0$, this proves that A satisfies (K_l).

3. **The main theorem.** A right ideal A of S is called large if $A \cap B \neq 0$ for every right ideal $B \neq 0$.

LEMMA 3.1. *Let S be a J_r-ring, and T an extension ring. Denote by U the set of all elements x of T such that $\{y \mid y \in S,\ xy \in S\}$ is a large right ideal of S. If every nonzero left ideal of T has a nonzero intersection with S, then U is a right quotient ring of S.*

PROOF. It is easily seen that U forms a subring of T. Let V be the set of all elements $y \in T$ such that $r(S, y)$ is large. Then V is a left ideal of T, and $V \cap S = 0$ since S is a J_r-ring. Hence $V = 0$ by the assumption. It follows from this that U is a right quotient ring of S, as desired.

We proved in [3] the following

THEOREM 3.2. *Every regular left self injective ring S is decomposed into the direct sum of two ideals A and B in such a way that A is strongly regular, and B is generated by idempotents.*

In fact, by [3, Theorem 4] $S = A \oplus B$, where A is strongly regular and B does not contain any nonzero strongly regular ideals. B is generated by idempotents by [3, Theorem 2].

By virtue of this theorem we obtain

THEOREM 3.3. *Let S be a J-ring. Then $\bar{S}_l = \bar{S}_r$ if and only if S satisfies (K_l) and (K_r).*

PROOF. Let $\bar{S}_l = \bar{S}_r$. By Corollary 2.3 $\bar{S}_l$ satisfies (K_l). Hence S also satisfies (K_l) by (ii) of Corollary 2.5. Similarly S satisfies (K_r), proving the only if part of the theorem. To see the if part let U be the set of all elements x of $\bar{S}_l$ such that $\{y \mid y \in S, xy \in S\}$ is a large right ideal of S. By Lemma 3.1 U is a right quotient ring of S. Let $\bar{S}_l \ni e = e^2$. Then $(\bar{S}_l e \cap S) \cap (\bar{S}_l(1-e) \cap S) = 0$. By Corollary 2.4 $L(S)$ is dually isomorphic to $R(S)$ and the correspondence is given by the annihilator relation. Since $\bar{S}_l e \cap S$, $\bar{S}_l(1-e) \cap S \in L(S)$, we have $r(S, \bar{S}_l e \cap S) \cup r(S, \bar{S}_l(1-e) \cap S) = S$ in $R(S)$. By Lemma 2.1

$r(S, \overline{S}_l e \cap S) = r(S, \overline{S}_l e) = r(\overline{S}_l, \overline{S}_l e) \cap S = (1-e)\overline{S}_l \cap S$, and similarly $r(S, \overline{S}_l(1-e) \cap S) = e\overline{S}_l \cap S$. Therefore $((1-e)\overline{S}_l \cap S) \cup (e\overline{S}_l \cap S) = S$. This implies that $((1-e)\overline{S}_l \cap S) + (e\overline{S}_l \cap S)$ is a large right ideal of S. Evidently $e(((1-e)\overline{S}_l \cap S) + (e\overline{S}_l \cap S)) = e\overline{S}_l \cap S \subset S$. It follows from this that $e \in U$. Therefore we have proved that every idempotent in $\overline{S}_l$ is contained in U. By Theorem 3.2 there exists a central idempotent f of $\overline{S}_l$ such that $\overline{S}_l f$ is strongly regular, and $\overline{S}_l(1-f)$ is generated by idempotents. Since $f \in U$, $U = Uf \oplus U(1-f)$. As is easily seen $\overline{S}_l f$ is the maximal left quotient ring of Uf. Since $\overline{S}_l f$ is strongly regular, $L(Uf)$ is distributive by Corollary 1.3. Now by (i) of Corollary 2.5 U satisfies (K_l) and (K_r), and hence Uf also satisfies them by Lemma 2.6, whence $L(Uf)$ and $R(Uf)$ are dually isomorphic by Corollary 2.4. Thus, $R(Uf)$ is distributive, and so the maximal right quotient ring of Uf is strongly regular by Corollary 1.3. Therefore $\overline{S}_l f$ is a right quotient ring of Uf by Theorem 1.4. On the other hand, since $\overline{S}_l(1-f)$ is generated by idempotents, and since U contains every idempotent in $\overline{S}_l$, it follows that $\overline{S}_l(1-f) = U(1-f)$. Hence $\overline{S}_l = \overline{S}_l f \oplus \overline{S}_l(1-f)$ is a right quotient ring of $U = Uf \oplus U(1-f)$. Since U is a right quotient ring of S, we see that $\overline{S}_l$ is a right quotient ring of S. We can prove similarly that $\overline{S}_r$ is a left quotient ring of S. Thus, we have $\overline{S}_l = \overline{S}_r$, completing the proof.

4. **WR-rings and continuous rings.** A ring is called a semisimple I-ring if every one-sided ideal contains a nonzero idempotent. By [2, (4, 10)] every semisimple I-ring is a J-ring. A ring is said to be of bounded index if there is an integer n such that $x^n = 0$ for every nilpotent element x. A semisimple I-ring is WR (=weakly reducible) if and only if every nonzero ideal contains a nonzero ideal of bounded index by [4, Theorem 9]. In [2, Theorem 5] we proved that $\overline{S}_l = \overline{S}_r$ for any semisimple WR-ring S. Therefore from Theorem 3.3 we obtain

THEOREM 4.1. *If a semisimple I-ring S is WR, then S has the following property: The right (left) annihilator of a left (right) ideal A is nonzero if and only if there exists a nonzero left (right) ideal B such that $A \cap B = 0$.*

Let S be a regular ring. If the lattice L of principal left ideals is complete, we say that S is a complete regular ring. A complete regular ring S is called continuous if L has the following property: $(\bigcup a_\alpha) \cap b = \bigcup (a_\alpha \cap b)$ and $(\bigcap a_\alpha) \cup b = \bigcap (a_\alpha \cup b)$ for any chain (a_α) and any element b in L. We proved in [4, Theorem 7] that a complete regular ring is continuous if and only if it satisfies (K_l) and (K_r). Thus, by Theorem 3.3 we obtain

THEOREM 4.2. *A complete regular ring* S *is continuous if and only if* $\overline{S}_l = \overline{S}_r$.

By [3, Theorem 3] every continuous regular ring is (both left and right) self injective if it contains no nonzero strongly regular ideals. Hence we have

THEOREM 4.3. *Let* S *be a complete regular ring, and suppose that* S *does not contain any nonzero strongly regular ideals. Then* $\overline{S}_l = \overline{S}_r$ *(if and) only if* S *is self injective, that is,* $S = \overline{S}_l = \overline{S}_r$.

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INVERSE LIMITS OF SOLVABLE GROUPS

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In this paper we generalize to groups of Galois type some results of P. Hall on finite solvable groups [1; 2; 3]. We need, in a modified form, some results of van Dantzig: the definition of supernatural numbers (which are related to van Dantzig's universal numbers) and Theorem 5, which he proved for ordinary p-Sylow subgroups [6]. Lemmas 1 and 4 and the method of proof in Theorem 5 are due to Tate [5].

A topological group G is of *Galois type* if it is compact and totally disconnected. In any Galois type group the open normal subgroups form a neighborhood base at the identity. Every closed subgroup is the intersection of the open subgroups containing it [4]. Whenever M and N are open normal subgroups of G and $N \supset M$ we shall write ϕ_N^M for the natural homomorphism of G/M onto G/N (these quotient groups are finite) and ϕ_N for the natural homomorphism of G onto G/N. G is the inverse limit of the groups $\{G/N\}$, N ranging over the open normal subgroups of G. Conversely, the inverse limit of finite groups is of Galois type.

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