

# A NOTE ON FINITE GROUPS WITH AN ABELIAN SYLOW GROUP

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It has been conjectured that if the order  $g$  of a finite noncyclic simple group  $G$  is divisible by a prime power  $p^n$ , then  $g > p^{2n}$ . We shall show this in the case that the  $p$ -Sylow groups of  $G$  are abelian. In fact, we shall prove the theorem:

**THEOREM.** *Let  $G$  be a finite group. Let  $p$  be a prime and assume that the  $p$ -Sylow subgroups  $P$  of  $G$  are abelian. Then the intersection of all  $p$ -Sylow subgroups of  $G$  appears as intersection of  $P$  with one of its conjugates.*

**COROLLARY.** *If the finite group  $G$  of order  $g$  has an abelian  $p$ -Sylow group  $P$  of order  $p^n$ ,  $g \neq p^n$ , and if the maximal normal  $p$ -subgroup  $D$  has order  $p^d$ , the number of conjugates of  $P$  is at least  $p^{n-d} + 1$  and  $g \geq p^n(p^{n-d} + 1)$ . In particular, if  $D = \{1\}$ ,  $g \geq p^n(p^n + 1)$ .*

**PROOF OF THE THEOREM.** If the  $p$ -Sylow group  $P$  has order 1, the theorem is trivial. We use induction. If  $D \neq \{1\}$ , we can deduce the statement for  $G$  from that for  $G/D$ . Hence we may assume  $D = \{1\}$ . Suppose that

$$(1) \quad P \cap P_1 \cap \cdots \cap P_r = \{1\}$$

where  $P_1, P_2, \dots, P_r$  are conjugates of the  $p$ -Sylow subgroup  $P = P_0$  of  $G$  and where the representation of  $\{1\}$  as such an intersection with a minimal  $r$  is chosen. If  $r = 1$ , we are finished. Assume then  $r \geq 2$  and set

$$(2) \quad M = P_1 \cap P_2 \cap \cdots \cap P_{r-1},$$

$$(3) \quad T = P \cap M = P \cap P_1 \cap \cdots \cap P_{r-1} \neq \{1\}$$

while by (1)

$$T \cap P_r = \{1\}.$$

Let  $H$  be the subgroup generated by  $P, P_1, \dots, P_{r-1}$ ,

$$H = \{P, P_1, P_2, \dots, P_{r-1}\}.$$

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Since the  $P_j$  are abelian and  $T \subseteq P_j$  for  $j=0, 1, \dots, r-1$ ,  $T$  is included in the center  $Z(H)$  of  $H$ . Hence  $H \neq G$ , since otherwise  $T$  would be normal in  $G$  and then  $T$  would belong to all  $p$ -Sylow subgroups of  $G$ , contrary to (1). In particular, the theorem is true for  $H$  in the place of  $G$ .

The group  $T(H \cap P_r)$  is an abelian  $p$ -subgroup of  $H$ . Let  $R$  be a  $p$ -Sylow subgroup of  $H$  with  $R \supseteq T(H \cap P_r)$ . By the theorem, applied to  $H$ , there exists a  $p$ -Sylow subgroup  $R^*$  of  $H$  such that the intersection  $X$  of all the  $p$ -Sylow subgroups of  $H$  is equal to  $R \cap R^*$ . The groups  $P, P_1, \dots, P_{r-1}$  are  $p$ -Sylow subgroups of  $H$  and hence

$$X \subseteq P \cap P_1 \cap \dots \cap P_{r-1} = T$$

while the  $p$ -group  $T \subseteq Z(H)$  must belong to all  $p$ -Sylow groups of  $H$  and hence to  $X$ . Thus,  $T = X$  and

$$T = R \cap R^*.$$

If  $\sigma \in R^* \cap P_r$ , then  $\sigma \in H \cap P_r \subseteq R$ . Hence  $\sigma \in R \cap R^* = T$ . Consequently,  $\sigma \in T \cap P_r = \{1\}$ ; cf. (3) and (1). This shows that  $R^* \cap P_r = \{1\}$ . Since  $P \subseteq H$ , the  $p$ -Sylow-groups of  $H$  are Sylow groups of  $G$ . In particular,  $R^*$  is a conjugate of  $P$ . Replacing  $P_r$  by a suitable conjugate, we obtain a  $p$ -Sylow subgroup  $P^\#$  of  $P$  with  $P \cap P^\# = \{1\}$  and this concludes the proof.

If in the notation of the theorem, we have  $D = P \cap P^*$  where  $P^*$  is a conjugate of  $P$ , then as is well known, we have  $p^{n-d}$  distinct  $p$ -Sylow groups of the form  $\sigma^{-1}P\sigma$  with  $\sigma \in P^*$ . Since they are all different from  $P^*$ , the number of conjugates of  $P$  is at least  $p^{n-d} + 1$ . On the other hand, the number of conjugates is  $g/m$  where  $m$  is the order of the normalizer of  $P$  in  $G$ . Since  $m \geq p^n$ ,  $g \geq p^n(p^{n-d} + 1)$ . This establishes Corollary 1.

**COROLLARY 2.** *If the finite group  $G$  of order  $g$  has an abelian  $p$ -Sylow group  $P$  of order  $p^n$ , if  $G$  possesses neither a normal  $p$ -subgroup different from  $\{1\}$  nor a normal subgroup of index  $p$ , then  $g \geq 2p^n(p^n + 1)$ .*

Indeed, it follows from Burnside's theorem that the normalizer of  $P$  has at least order  $2p^n$  in this case while the number of  $p$ -Sylow groups is at least  $p^n + 1$ .

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