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A NOTE ON SUBGROUPS OF THE MODULAR GROUP[1]

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1. We will follow the notation of [4]. Let Γ denote the 2×2 modular group, that is, the set of all 2×2 matrices with rational integral entries and determinant 1. For each positive integer m define $\Gamma(m)$, *the principal congruence subgroup of level m*, by

$$\Gamma(m) = \left\{X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, a \equiv d \equiv 1,\ b \equiv c \equiv 0 \pmod{m}\right\}.$$

Let

$$T_m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

and let $\Delta(m)$ be the normal subgroup of Γ generated by T_m. That is, $\Delta(m)$ is the smallest normal subgroup of Γ containing T_m. Clearly, $\Delta(m) \subset \Gamma(m)$.

In [4] Reiner considers the following questions raised in [1]:

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(a) Does $\Delta(m)=\Gamma(m)$ for all m?

(b) For each m, does there exist a positive integer k such that $\Delta(m)\supset\Gamma(mk)$?

He answers both questions in the negative by proving that *if* $m>1$ *and* m *is not a prime power, then* $\Delta(m)$ *does not contain any principal congruence subgroup*. The situation when m is a prime power is left open.

The purposes of this note are to point out that the following result is to be found (at least implicitly) in [2] and to give a new proof.

THEOREM. *If* $m\geqq 6$, *then* $\Delta(m)$ *is of infinite index in* $\Gamma(m)$. *Since a principal congruence is always of finite index in* Γ, *it is a consequence that for* $m\geqq 6$, $\Delta(m)$ *contains no principal congruence subgroup.*

I would like to thank Dr. J. R. Smart for calling my attention to [4].

We will make use of the following simple

LEMMA. $\Delta(m)$ *is generated by the set*

$$H=\{X^{-1}T_mX \mid X\in\Gamma\}.$$

The proof is obtained by noting that the group G generated by H is normal in Γ and that $G\subset\Delta(m)$.

Using this lemma we obtain from [2, pp. 267, 354–356] that $\Delta(m)=\Gamma(m)$, for $1\leqq m\leqq 5$, and from [2, pp. 356–360] that $\Delta(m)$ is of infinite index in $\Gamma(m)$, for $m\geqq 6$.

2. We now give an independent proof of this latter fact based upon the results of [3]. In [3] it was shown that given $m\geqq 2$ there exists a function, say $\lambda_1(m;\tau)$, defined and analytic in $\mathfrak{I}(\tau)>0$ such that

(i) $\lambda_1(m;X\tau)=\lambda_1(m;\tau)+c(X)$, for each $X\in\Gamma(m)$ and for $\mathfrak{I}(\tau)>0$, where $c(X)$ is independent of τ;

(ii) $c(X)=0$, when

$$X=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

is parabolic, that is, when $|a+d|=2$;

(iii) $\lambda_1(m;\tau)$ has a pole of order 1 in the local uniformizing variable at one of the parabolic cusps of $\mathfrak{F}_m$, the fundamental region of $\Gamma(m)$, and is regular at all of the other parabolic cusps of $\mathfrak{F}_m$.

Of course, (i) and (iii) show that $\lambda_1(m;\tau)$ is an *abelian integral* connected with $\Gamma(m)$.

Now T_m is parabolic and a simple computation shows that each element of H is parabolic. But $c(X_1X_2)=c(X_1)+c(X_2)$ and by (ii) $c(X)=0$, for $X\in\Delta(m)$. Thus for $X\in\Gamma(m)$, $c(X)$ depends only upon the coset of X modulo $\Delta(m)$. Thus if $\Delta(m)$ has finite index in $\Gamma(m)$, there are only finitely many distinct values $c(X)$, with $X\in\Gamma(m)$. But this implies that $c(X)=0$ for all $X\in\Gamma(m)$. For if there exists $X_0\in\Gamma(m)$, with $c(X_0)\neq 0$, then $c(X_0^t)=tc(X_0)$; $t=1, 2, 3, \cdots$ provides us with infinitely many distinct values $c(X)$.

Thus if $\Delta(m)$ has finite index in $\Gamma(m)$, $\lambda_1(m;\tau)$ is an invariant with respect to $\Gamma(m)$ with precisely one pole of order 1 in $\mathfrak{F}_m$. Since $\mathfrak{F}_m$ is compact, the Riemann-Roch Theorem implies that the genus of $\mathfrak{F}_m$ is zero. By [2, p. 398] $\mathfrak{F}_m$ has genus zero exactly when $1\leqq m\leqq 5$. Thus for $m\geqq 6$, $\Delta(m)$ has infinite index in $\Gamma(m)$.

The proof shows that when $m\geqq 6$, $\lambda_1(m;\tau)$ cannot be invariant with respect to any subgroup of finite index in $\Gamma(m)$.

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