# FUNCTION ALGEBRAS WITH CLOSED RESTRICTIONS[1]

I. GLICKSBERG

Let $C(X)$ be the (supremum normed) algebra of all complex continuous functions on a compact Hausdorff space $X$, and $A$ a closed subalgebra containing the constants and separating the points of $X$. For $F \subset X$ let $f|F$ be the restriction of the function $f$ to $F$, $A|F = \{f|F : f \in A\}$. The purpose of this note is to prove the following result, which characterizes $C(X)$ among its closed separating subalgebras.

THEOREM. *Suppose that for every closed $F \subset X$, $A|F$ is closed in $C(F)$. Then $A = C(X)$.*

Our proof leans heavily on some facts obtained recently by Katznelson [1]. In fact little else, beyond the Riesz representation theorem, is needed (for some facts concerning Banach algebras the reader is referred to [2; 3]).

Let $F$ be a closed subset of $X$. $kF = \{f : f \in A, f(F) = 0\}$, the kernel of $F$, is an ideal in $A$, and we can form the quotient algebra $A/kF$; for an element $f + kF (f \in A)$ of $A/kF$, $\|f + kF\|$ will denote the usual quotient norm ($\inf \{\|f+g\| : g \in kF\}$) under which $A/kF$ is a Banach algebra. For $x \in F$, $f + kF \to f(x)$ is a multiplicative linear functional on $A/kF$, so necessarily of norm $\leqq 1$, whence $\|f|F\| = \sup |f(F)| \leqq \|f + kF\|$, where $\|f|F\|$ is of course the norm inherited by the closed subalgebra $A|F$ of $C(F)$. Now clearly $f + kF \to f|F$ is an isomorphism of $A/kF$ onto $A|F$, and so, by the open mapping theorem, topological: there is a constant $c_F$ for which

$$\|f + kF\| \leqq c_F \|f|F\|, \qquad f \in A, \tag{1}$$

a relation which we shall need somewhat later.

To begin the proof proper we shall show that for closed disjoint subsets $F$ and $K$ of $X$ there is an $f$ in $A$ which is 1 on $F$, 0 on $K$. Indeed let $x$ and $y$ be distinct elements of $X$. Since $A$ separates $X$, there is an $f$ in $A$ with $f(x) = 0, f(y) = 1$, so we can find closed neighborhoods $V_x$ of $x$ and $W_x$ of $y$ for which $|f(V_x)| \leqq \frac{1}{4}$, $|1 - f(W_x)| \leqq \frac{1}{4}$. As is well known there is a sequence $\{p_n\}$ of polynomials in one complex variable which converge uniformly on $\{z : |z| \leqq \frac{1}{4}\} \cup \{z : |1 - z| \leqq \frac{1}{4}\}$ to the characteristic function of $\{z : |z| \leqq \frac{1}{4}\}$; consequently

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$\{(p_n \circ f)\,|\,(V_x \cup W_x)\}$ is a sequence of elements of $A\,|\,(V_x \cup W_x)$ which converges uniformly on $V_x \cup W_x$, tending to 1 on $V_x$, 0 on $W_x$. Since $A\,|\,(V_x \cup W_x)$ is closed under uniform convergence by hypothesis, there is an $e$ in $A$ which is 1 on $V_x$, 0 on $W_x$.

Now let $y$ be a fixed element of $K$. $F$ can be covered by finitely many $V_x$, $x \in F$, say $V_{x_1}, \cdots, V_{x_n}$. If $e_1, \cdots, e_n$ are the corresponding elements of $A$, then (as in a familiar argument) clearly $e_1' = e_1 + e_2 - e_1 e_2$ is 1 on $V_{x_1} \cup V_{x_2}$ and 0 on the neighborhood $W_{x_1} \cap W_{x_2}$ of $y$; similarly $e_2' = e_1' + e_3 - e_1' e_3$ is 1 on $V_{x_1} \cup V_{x_2} \cup V_{x_3}$ and 0 on $W_{x_1} \cap W_{x_2} \cap W_{x_3}$. Continuing we thus obtain an element $f_y$ of $A$ which is 1 on $F$, and vanishes on some neighborhood $U_y$ of $y$.

But now finitely many $U_y$ cover $K$, so that the product $f$ of the corresponding $f_y$ yields an element of $A$ which is 1 on $F$ and 0 on $K$, as desired.

Alternatively stated, we have found an idempotent in $A/k(K \cup F)$ which separates $F$ and $K$. What we now require is a bound on the norms of such idempotents, and to obtain this we repeat an argument of Katznelson's [1, Lemmas 3, 4, 5]. Let us say *$A$ is bounded on a subset $V$ of $X$* if there is a constant $C_V$ for which, whenever $F$ is closed and $F \subset V$, any idempotent in $A/kF$ has norm $< C_V$. (Our next lemmas are taken from Katznelson [1], and are repeated here for the convenience of the reader.)

LEMMA 1. *Let $V_1$ and $V_2$ be open in $X$, and $A$ be bounded on each $V_i$. Then $A$ is bounded on every closed subset $F$ of $V_1 \cup V_2$.*

PROOF. Since $F \backslash V_2$ is a closed subset of $V_1$ we can find an open $W_1 \supset F \backslash V_2$, $\overline{W_1} \subset V_1$; and since $F \backslash W_1 \subset V_2$ there is an open $W_2 \supset F \backslash W_1$, $\overline{W_2} \subset V_2$. $F \backslash W_1$ and $F \backslash W_2$ are disjoint closed subsets of $X$ so there is a $\phi$ in $A$ for which $\phi(F \backslash W_1) = 0$, $\phi(F \backslash W_2) = 1$, as we have seen.

Let $P$ be a closed subset of $F$ and let $f + kP$ be an idempotent in $A/kP$, so $f$ is zero or 1 at each point of $P$. Then $f + k(P \cap \overline{W_i})$ is an idempotent in $A/k(P \cap \overline{W_i})$ and so (by the definition of the quotient norm) we can choose $f_i \in f + k(P \cap \overline{W_i})$ with $\|f_i\| < C_{V_i}$, $P \cap \overline{W_i}$ being a closed subset of $V_i$, $i = 1, 2$. But now $f = \phi f_1 + (1 - \phi) f_2$ on $P$, since $P \subset F \subset W_1 \cup W_2$, and thus

$$\|f + kP\| \leqq \|\phi f_1 + (1 - \phi) f_2\| < \|\phi\| C_{V_1} + \|1 - \phi\| C_{V_2}$$

so the right side may be taken as $C_F$.

Still following Katznelson let us say *$A$ is bounded at $x \in X$* if $A$ is bounded on $V$, for some neighborhood $V$ of $x$. Then as an obvious consequence of Lemma 1 and compactness we have

LEMMA 2. *If $F$ is closed and $A$ is bounded at each $x$ in $F$ there is an*

*open $V \supset F$ on which $A$ is bounded.*

LEMMA 3. *There are at most finitely many $x$ in $X$ at which $A$ is not bounded.*

PROOF. If there were infinitely many we could find a sequence $\{x_n\}$ of them with disjoint open neighborhoods $\{V_n\}$ for which $A$ is not bounded on $V_n$. So each $V_n$ would contain a closed $F_n$ for which there is an idempotent $f_n + kF_n$ in $A/kF_n$ of norm $\geqq n$.

Now let $F = (\bigcup F_n)^-$; since $V_n \cap F_m = \emptyset$ for $n \neq m$, $F = F_n \cup (\bigcup_{m \neq n} F_m)^-$, $F_n \cap (\bigcup_{m \neq n} F_m)^- = \emptyset$ and thus there is a $\phi$ in $A$ with $\phi(F_n) = 1$, $\phi(F \backslash F_n) = 0$. Consequently $\phi f_n$ is zero or 1 at each $x$ in $F$, and $\phi f_n + kF$ is an idempotent in $A/kF$; but

$$n \leqq \|f_n + kF_n\| = \|\phi f_n + kF_n\| \leqq \|\phi f_n + kF\|$$

(since $kF \subset kF_n$) so the idempotents in $A/kF$ are not bounded in norm. By (1) they must be bounded, and the lemma is proved.

Let $\{x_1, \cdots, x_n\}$ be the finite set given by Lemma 3 and let $F$ be any closed set in $X$ disjoint from $\{x_1, \cdots, x_n\}$. By Lemma 2, $A$ is bounded on a neighborhood of $F$; so there is a constant $C_F$ for which, whenever $K$ is closed, $K \subset F$, each idempotent in $A/kK$ has norm $< C_F$.

Now suppose $\mu$ is a (complex regular Borel) measure on $F$ orthogonal to the closed subalgebra $A \mid F$ of $C(F)$, and let $K$ be a closed subset of $F$, $\epsilon > 0$. Choose a closed $K_0 \subset F \backslash K$ for which the total variation of $\mu$ on $(F \backslash K) \backslash K_0$ is less than $\epsilon / C_F$, and then choose $f \in A$ which is 1 on $K$, 0 on $K_0$, and of norm $< C_F$. Then

$$0 = \int f d\mu = \int_K 1 d\mu + \int_{K_0} 0 d\mu + \int_{(F \backslash K) \backslash K_0} f d\mu,$$

so $|\mu(K)| = |\int_{(F \backslash K) \backslash K_0} f d\mu| \leqq C_F \cdot \epsilon / C_F = \epsilon$. Thus since $\epsilon > 0$ is arbitrary, $\mu K = 0$ for every closed $K \subset F$, and $\mu$ itself vanishes by regularity. Consequently $A \mid F = C(F)$.

Now this implies any measure $\mu$ orthogonal to $A$ is carried by $\{x_1, \cdots, x_n\}$. For if not $\mu F \neq 0$ for some closed $F$ disjoint from $\{x_1, \cdots, x_n\}$ by regularity, and if $V$ is an open neighborhood of $\{x_1, \cdots, x_n\}$ with $V^- \cap F = \emptyset$, we can choose $f \in A$ with $f(V^-) = 0$, $f(F) = 1$; then $f\mu$ (the measure corresponding to the functional $g \to \int g f d\mu$ on $C(X)$) is a measure carried by $X \backslash V$, and orthogonal to $A$ and so orthogonal to $A \mid (X \backslash V) = C(X \backslash V)$, hence the zero measure. But $f\mu(F) = \int_F f d\mu = \mu F \neq 0$, the desired contradiction, and $\mu$ orthogonal to $A$ must be carried by $\{x_1, \cdots, x_n\}$.

Since $A$ separates the points of $X$ this is clearly impossible unless all orthogonal measures vanish, i.e., unless $A = C(X)$, which completes the proof of the theorem.

## References

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Institute for Advanced Study