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VANISHING CENTRAL DIFFERENCES

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Given a sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers, a number of theorems have been proved concerning implications of the vanishing of certain differences $\Delta^n a_0$ if the given sequence satisfies some growth restriction. The first such result, proved by Agnew [1], states that if $\{a_n\}$ is bounded and $\Delta^{2n}a_0=0$ for all n, then $a_n=0$ for all n. If all the odd differences are zero, the sequence is constant. Fuchs [6] proved the following: Let $a_n=o(n^k)$ for some positive number k, and let n_j be a subsequence of the positive integers such that if $n(R)$ is the number of $n_j<R$, then $n(R)\geqq R/2$ for $R>R_0$. If $\Delta_j^n a_0=0$ for all n_j, then $a_n=p(n)$ where $p(x)$ is a polynomial of degree less than k. Buck [4] assumed only that lim sup $|a_n|^{1/n}<1$ and $\Delta^n a_0=0$ for all n belonging to a set of positive integers of density $d>\frac{1}{2}$ and proved there is a function f of exponential type whose growth function $h(\theta)$ satisfies $h(\pm\pi/2)<\pi$ such that $f(n)=a_n$ for all n. In this paper, we show that if the given sequence is extended to $\{a_n\}_{n=-\infty}^{\infty}$ by letting $a_{-n}=a_n$, then the vanishing of certain of the even central differences $\Delta^{2n}a_{-n}$ has similar implications. Or, letting $a_{-n}=-a_{n-1}$, vanishing of odd differences $\Delta^{2n-1}b_{-n}$ gives similar results.

If G is a connected set, let $K[G]$ denote the class of all entire functions of exponential type whose conjugate indicator diagrams $D(f)$ are contained in G. If G is the rectangle $\{x+iy\,|\,|x|\leqq a;\ |y|\leqq c\}$, then $K[a,c]$ will be used for $K[G]$. Let $C_{z,n}$ denote the polynomial $z(z-1)\cdots(z-n+1)/n!$.

Certain results concerning the sequence $\{\mathfrak{L}_n\}$ of Stirling functionals given by $\mathfrak{L}_n(f)=\Delta^n f(-n/2)$ will be needed. These functionals

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have the representation

$$\Delta^n f(-n/2) = \frac{1}{2\pi i}\int_\Gamma (e^{\zeta/2} - e^{\zeta/2})^n F(\zeta)d\zeta$$

where F is the Borel (Laplace) transform of f and Γ is any simple contour enclosing the conjugate indicator diagram $D(f)$. Let B be the set of all ζ satisfying $|e^{\zeta/2}-e^{\zeta/2}|<2$. Then B is a convex, lens-shaped region, symmetric about the origin whose boundary has vertices at $\pm\pi i$ and crosses the real axis at $\pm\log(3+2\sqrt{2})$. For f in $K[B]$, Buck [3] showed that

$$f(z) = \sum_{n=0}^{\infty} \Delta^n f(-n/2)(z/n)C_{z+n/2-1,n-1},$$

convergent for all z. The author [5] showed that for a given sequence $\{c_n\}$ of complex numbers, there is a function f in $K[B]$ such that $\Delta^n f(-n/2)=c_n$; $n=0, 1, 2, \cdots$ if and only if $\limsup |c_n|^{1/n}<2$. If we let $G(t)=\sum c_n t^n$, then f has the representation

$$f(z) = \frac{1}{2\pi i}\int_E \frac{G(t)}{t}\exp[2z\sinh^{-1} 1/(2t)]dt \tag{1}$$

where E is a simple contour contained in the region of regularity of G and enclosing the disk $|t|\leq\frac{1}{2}$. Then the conjugate indicator diagram $D(f)$ is contained in the convex hull of the image of E under the map $\zeta=2\sinh^{-1} 1/(2t)$.

THEOREM 1. *Let* $\{b_n\}_{n=-\infty}^{\infty}$ *be an even sequence of complex numbers such that* $\limsup |b_n|^{1/n}\leq 1$. *Suppose there is a set A of positive integers of density $d>0$ such that for all n in A,* $\Delta^{2n}b_{-n}=0$. *Then* $\sum\Delta^{2n}b_{-n}(z/2n)C_{z+n-1,n-1}$ *converges to an even function f in $K[B]$ and* $f(n)=b_n$; $n=0, \pm 1, \pm 2, \cdots$.

We need the following lemma.

LEMMA. *For a sequence* $\{b_n\}_{n=-\infty}^{\infty}$, *define $Q(t)$ formally by*

$$Q(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} C_{n,k}b_{-n+2k}t^n$$

and define $P(z)$ by $P(z)=(1+2z)^{-1}Q(z/(1+2z))$. *Then formally* $P(z) = \sum\Delta^{2n}b_{-n}z^n$.

PROOF.[1] Let E be an operator defined for a sequence $a=\{a_k\}$ by $E(a)(k)=a(k+1)$. Then $\Delta=E-1$, and we have

[1] This proof was suggested by Professor R. C. Buck.

$$\begin{aligned}
Q(t) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} C_{n,k}b_{-n+2k}t^n \\
&= \sum_{n=0}^{\infty} t^n E^{-n}\sum_{k=0}^{n} C_{n,k}E^{2k}b_0 \\
&= \sum_{n=0}^{\infty} t^n(E+E^{-1})^n b_0 \\
&= \frac{1}{1-t(E+E^{-1})}\,b_0\,. \\
Q(z/(1+2z)) &= (1+2z)\frac{1}{1-z(E-2+E^{-1})}\,b_0 \\
&= (1+2z)\sum_{n=0}^{\infty} z^n(E-2+E^{-1})^n b_0 \\
&= (1+2z)\sum_{n=0}^{\infty} \Delta^{2n}b_{-n}z^n. \qquad \text{Q.E.D.}
\end{aligned}$$

Proof of Theorem 1. Since

$$\limsup |b_n|^{1/n} \leqq 1, \quad \limsup\left|\sum_{k=0}^{n} C_{n,k}b_{-n+2k}\right| \leqq 2.$$

Thus $Q(t)$ is regular in the disk $|t|<\frac{1}{2}$. Then from its definition $P(z)$ is regular for $|z/(1+2z)|<\frac{1}{2}$ or $|z|<|z+\frac{1}{2}|$ which is the set of all z whose real part is greater than $-\frac{1}{4}$. Let $G(z)=P(z^2)=\sum\Delta^{2n}b_{-n}z^{2n}$. Then G is regular for all z such that $\Re(z^2)>-\frac{1}{4}$; i.e., for z in the region containing the origin and bounded by the equilateral hyperbola $y^2-x^2=\frac{1}{4}$ where $z=x+iy$. Let r_0 be the radius of convergence of $\sum\Delta^{2n}b_{-n}z^{2n}$. Then $r_0\geqq\frac{1}{2}$. Since G is even, $G(z)=\sum c_nz^n$ where $c_{2n+1}=0$ and $c_{2n}=\Delta^{2n}b_{-n}$. Then from the hypothesis, $c_n=0$ for all n belonging to a set of density $d'>\frac{1}{2}$. Thus, by Pólya's density theorem [7], G has a singularity on every arc of $|z|=r_0$ of opening $2\pi(1-d)$ and this is less than π. But if $r_0=\frac{1}{2}$, the only possible singularities are at $i/2$ or $-i/2$; so $r_0>\frac{1}{2}$. Then, by the results on Stirling functionals quoted earlier, there is a function f in $K[B]$ such that $\Delta^{2n}f(-n)=\Delta^{2n}b_{-n}$ for all n, and $f(z)=\sum\Delta^{2n}b_{-n}(z/2n)C_{z+n-1,n-1}$ convergent for all z. Since $(z/2n)C_{z+n-1,n-1}$ is even for each n, f is even. It can be shown by induction that $f(n)=b_n$; $n=0,\ \pm1,\ \pm2,\cdots$, using the fact that $\Delta^{2n}f(-n)=\Delta^{2n}b_{-n}$ for each n. Q.E.D.

Theorem 2. *In Theorem* 1, *if* $d\geqq\frac{1}{2}$, *then* f *is of zero type.*

Proof. If $d\geqq\frac{1}{2}$, then $G(z)=\sum\Delta^{2n}b_{-n}z^{2n}$ has zero coefficients for all n belonging to a set of positive integers of density at least $\frac{3}{4}$. Then, by Pólya's density theorem, G has a singularity on every arc of its circle of convergence of opening $\pi/2$. But G is regular for all $z=re^{i\theta}$ with $\theta \leqq \pi/4$, so G is entire. Then in representation (1) of f, the contour E can be taken as a circle of arbitrarily large radius, so that its image under the map $\zeta=2\sinh^{-1} 1/(2t)$ can be made to lie in an arbitrarily small disk about the origin. Therefore $D(f)$ is the origin, i.e., f is of zero type. Q.E.D.

COROLLARY 3. *In Theorem* 1, *if* $d \geqq \frac{1}{2}$ *and* $b_n = o(n^k)$ *as* $n \rightarrow \infty$ *for some* $k > 0$, *then* f *is a polynomial of degree less than* k.

PROOF. The function f is of zero type and since f is even, $f(n) = o(|n|^k)$ as $n \rightarrow \pm\infty$; so f is a polynomial of degree less than k [2, p. 183].

COROLLARY 4. *In Corollary* 3, *if* $\{b_n\}$ *is bounded then it is a constant sequence.*

Thus we have obtained theorems analogous to those of Buck, Fuchs, and Agnew referred to at the beginning.

Since, for an odd sequence $\{c_n\}_{n=-\infty}^{\infty}$, $\Delta^{2n} c_{-n} = 0$ for all n, we have the following:

COROLLARY 5. *If any bounded sequence* $\{b_n\}_{n=-\infty}^{\infty}$ *has* $b_0 = 0$ *and* $\Delta^{2n} b_{-n} = 0$ *for all* n *belonging to a set of positive integers of density* $d \geqq \frac{1}{2}$, *then* $\{b_n\}$ *is an odd sequence.*

For a sequence $\{b_n\}_{n=-\infty}^{\infty}$ such that $b_{-n} = -b_{n-1}$; $n = 1, 2, 3, \cdots$, we obtain theorems identical with those above except that the even differences are replaced by odd differences $\Delta^{2n-1} b_{-n}$ and the interpolating function is an odd function. The proofs of these theorems are almost the same as the proofs of the above theorems.

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